Derivatives
Pricing a Forward / Futures Contract

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Valuing forward contracts: Key ideas

- Two different ways to own a unit of the underlying asset at maturity:
  1. Buy spot (SPOT PRICE: $S_0$) and borrow => Interest and inventory costs
  2. Buy forward (AT FORWARD PRICE $F_0$)

- VALUATION PRINCIPLE: NO ARBITRAGE
- In perfect markets, no free lunch: the 2 methods should cost the same.

*You can think of a derivative as a mixture of its constituent underliers, much as a cake is a mixture of eggs, flour and milk in carefully specified proportions. The derivative’s model provide a recipe for the mixture, one whose ingredients’ quantity vary with time.*

Emanuel Derman, Market and models, *Risk* July 2001
Discount factors and interest rates

- Review: Present value of $C_t$
  - $PV(C_t) = C_t \times \text{Discount factor}$

- With annual compounding:
  - Discount factor $= \frac{1}{(1+r)^t}$

- With continuous compounding:
  - Discount factor $= \frac{1}{e^{rt}} = e^{-rt}$
Forward contract valuation: No income on underlying asset

- Example: Gold (provides no income + no storage cost)
  - Current spot price $S_0 = \$1,340/oz$
  - Interest rate (with continuous compounding) $r = 3\%$
  - Time until delivery (maturity of forward contract) $T = 1$

- Forward price $F_0$?

Strategy 1: buy forward

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$S_T - F_0$</td>
<td></td>
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</table>

Strategy 2: buy spot and borrow

<table>
<thead>
<tr>
<th></th>
<th>$t = 0$</th>
<th>$t = 1$</th>
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</thead>
<tbody>
<tr>
<td>Buy spot</td>
<td>-1,340</td>
<td>+ $S_T$</td>
</tr>
<tr>
<td>Borrow</td>
<td>+1,340</td>
<td>-1,381</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>$S_T - 1,381$</td>
</tr>
</tbody>
</table>

$F = \text{price "forward"}$
$f = \text{forward value}$

Should be equal
Forward price and value of forward contract

(Financial asset without any payout in the meantime)

Forward price:

\[ F_0 = S_0 e^{rT} \]

Remember: the forward price is the delivery price which sets the value of a forward contract equal to zero.

Value of forward contract with delivery price \( K \)

\[ f = S_0 - Ke^{-rT} \]

You can check that \( f = 0 \) for \( K = S_0 e^{rT} \)
Arbitrage

- If $F_0 \neq S_0 e^{rT}$: arbitrage opportunity

- Cash and carry arbitrage if $F_0 > S_0 e^{rT}$
  - Borrow $S_0$, buy spot and sell forward at forward price $F_0$

- Reverse cash and carry arbitrage if $S_0 e^{rT} > F_0$
  - Short asset, invest and buy forward at forward price $F_0$
Arbitrage: examples

- Gold – $S_0 = 1,750$, $r = 1.14\%$, $T = 1$  
  
  
  
  $S_0 e^{rT} = 1,770$

- If forward price = $1,800$
  - Buy spot $-1,750$
  - Borrow $+1,750$
  - Sell forward $0$
  - Total $0$

- If forward price = $1,700$
  - Sell spot $+1,750$
  - Invest $-1,750$
  - Buy forward $0$
  - Total $0$

  
  
  
  
  +30
Repurchase agreement (repo)

- An important kind of forward contract.
- Sale of a security + agreement to buy it back at a fixed price. (=reverse cash-and-carry, sale coupled with long forward)
- Underlying security held as a collateral
- Most repo are overnight
- Counterparty risk => haircut = Value of collateral - Loan
Schematic repo transaction

**Time t**
- **MARKET**
  - Buy bond at $Pt$
  - Pay $Pt$

**TRADER**
- Deliver bond
- Get $Pt$ - haircut

**REPO DEALER**
- Get bond

**Time T**
- **MARKET**
  - Sell bond at $PT$
  - Get $PT$

**TRADER**
- Get bond

**REPO DEALER**
- Pay $(Pt - \text{Haircut})$
  - $(1 + \text{RepoRate} \times (T-t))$
The repo market

Financial institution

Deposit (overnight)

Institutional investors
Non financial firms

Collateral
Market value + haircut

Huge repo market – exact size unknown

(No checking account insured by FDIC)
Spot rates and the yield curve

A yield curve is a representation of the relationship between market remuneration rates and the remaining time to maturity of debt securities, also known as the term structure of interest rates. The ECB publishes several yield curves as shown below.
Forward on a zero-coupon : example

- Consider a:
  - 6-month forward contract
  - on a 1-year zero-coupon with face value $A = 100$
  - Current interest rates (with continuous compounding)
    - 6-month spot rate: 4.00%
    - 1-year spot rate: 4.30%

- Step 1: Calculate current price of the 1-year zero-coupon
  - use 1-year spot rate
  - $S_0 = 100 \ e^{-(0.043)(1)} = 95.79$

- Step 2: Forward price = future value of current price
  - use 6-month spot rate
  - $F_0 = 95.79 \ e^{(0.04)(0.50)} = 97.73$
Forward on zero coupon

- Notations (continuous rates):
  \( A \equiv \text{face value of ZC with maturity } T^* \)
  \( r^* \equiv \text{interest rate from 0 to } T^* \)
  \( r \equiv \text{interest rate from 0 to } T \)
  \( R \equiv \text{forward rate from } T \text{ to } T^* \)
- Value of underlying asset:
  \( S_0 = A e^{-r^* T^*} \)
- Forward price:
  \( F_0 = S_0 e^r T \)
  \( = A e^{(rT - r^* T^*)} \)
  \( = A e^{-R(T^*-T)} \)
Forward interest rate

- Rate $R$ set at time 0 for a transaction (borrowing or lending) from $T$ to $T^*$
- With continuous compounding, $R$ is the solution of:
  \[ A = F_0 e^{R(T^* - T)} \]
  - The forward interest rate $R$ is the interest rate that you earn from $T$ to $T^*$ if you buy forward the zero-coupon with face value $A$ for a forward price $F_0$ to be paid at time $T$.
- Replacing $A$ and $F_0$ by their values:
  \[ R = \frac{r^* T^* - rT}{T^* - T} \]

In previous example:
\[ R = \frac{4.30\% \times 1 - 4\% \times 0.50}{1 - 0.50} = 4.60\% \]
Forward rate: Example

- Current term structure of interest rates (continuous):
  - 6 months: 4.00% ⇒ \( d = \exp(-0.040 \times 0.50) = 0.9802 \)
  - 12 months: 4.30% ⇒ \( d^* = \exp(-0.043 \times 1) = 0.9579 \)
- Consider a 12-month zero coupon with \( A = 100 \)
- The spot price is \( S_0 = 100 \times 0.9579 = 95.79 \)
- The forward price for a 6-month contract would be:
  - \( F_0 = 97.79 \times \exp(0.04 \times 0.50) = 97.73 \)
- The continuous forward rate is the solution of:
  - \( 97.73 e^{0.50R} = 100 \)
  ⇒ \( R = 4.60\% \)
Forward rate - illustration
Forward borrowing

- View forward borrowing as a forward contract on a ZC
  You plan to borrow $M$ for $\tau$ years from $T$ to $T^*$
  Define: $\tau = T^* - T$
  The *simple* interest rate set today is $R_S$
  You will repay $M(1+R_S\tau)$ at maturity
- In fact, you sell forward a ZC
  The face value is $M(1+R_S\tau)$
  The maturity is $T^*$
  The delivery price set today is $M$
- The interest will set the value of this contract to zero
Forward borrowing: Gain/loss

• At time $T^*$:
• Difference between the interest paid $R_s$ and the interest on a loan made at the spot interest rate at time $T$ : $r_s$

$$M\left[ r_s - R_s \right] \tau$$

• At time $T$:
  • $\Pi_T = \left[ M \left( r_s - R_s \right) \tau \right] / \left(1 + r_s \tau\right)$
FRA (Forward rate agreement)

- **Example: 3/9 FRA**
- *Buyer* pays *fixed* interest rate \( R_{fra} \) 5%
- *Seller* pays *variable* interest rate \( r_s \) 6-m LIBOR
- on *notional* amount \( M \) $100 m
- for a given time period (contract period) \( \tau \) 6 months
- at a future date (settlement date or reference date) \( T \) (in 3 months, end of accrual period)

- Cash flow for buyer (long) at time \( T \):
  - Inflow \( (100 \times \text{LIBOR} \times 6/12)/(1 + \text{LIBOR} \times 6/12) \)
  - Outflow \( (100 \times 5\% \times 6/12)/(1 + \text{LIBOR} \times 6/12) \)

- Cash settlement of the difference between present values
FRA: Cash flows 3/9 FRA (buyer)

\[ \frac{(100 \times 5\% \times 0.50)}{(1 + r_S 0.50)} \]

\[ \frac{(100 \times r_S 0.50)}{(1 + r_S 0.50)} \]

- General formula: \( CF = M[(r_S - R_{fra})\tau]/(1+r_S \tau) \)
- Same as for forward borrowing - long FRA equivalent to cash settlement of result on forward borrowing
Review: Long forward (futures)

Synthetic long forward:
borrow $S_0$ and buy spot

$$f_0 = 0$$

$$F_0 = S_0 e^{rT}$$

$$f_t = S_t - F_0 e^{-rT} = (F_t - F_0) e^{-rT}$$

Receive underlying asset

Pay forward price

Value of forward contract
Review Long forward on zero-coupon

(1) Face value of zero-coupon given

$$S_0 = Ae^{-r^*T}$$

$$A = F_0e^{R\tau}$$

(2) Calculate forward price which set the initial value of the contract equal to 0.

$$F_0 = S_0e^{rT}$$

$$f_t = S_t - F_0e^{-rT} = (F_t - F_0)e^{-rT}$$

(3) Calculate forward interest rate
Review Forward investment = buy forward zero-coupon

(1) Forward price given

\[ S_0 = A e^{-r^* T^*} \]

\[ f_0 = 0 \]

\[ S_T \]

\[ f_T = S_T - M = \frac{M (1 + R_S \tau)}{1 + r_S \tau} - M = \frac{M (R_S - r_s) \tau}{1 + r_s \tau} \]

(2) Calculate face value of zero-coupon which set the value of the contract equal to 0

\[ A = M (1 + R_S \tau) = F_0 e^{R \tau} \]
Review Forward borrowing = sell forward zero-coupon

\[ S_0 = A e^{-r^* T^*} \]

\[ f_0 = 0 \]

\[ S_0 = M - S_T = M - \frac{M(1 + R_S \tau)}{1 + r_s \tau} = \frac{M(r_s - R_S) \tau}{1 + r_s \tau} \]
Review Long Forward Rate Agreement

\[ S_0 = Ae^{-r_s \tau} \]

\[ f_T = \frac{M (r_s - R_{fra}) \tau}{1 + r_s \tau} \]

\[ \frac{M r_s \tau}{1 + r_s \tau} \]
To lock in future interest rate

- Borrow short & invest long
- Buy forward zero coupon
- Invest forward at current forward interest rate
- Sell FRA and invest at future (unknown) spot interest rate
Valuing a FRA

\[ \frac{Mr_s \tau}{1 + r_s \tau} + \frac{M}{1 + r_s \tau} = M \]

\[ f_t = M \times d(T - t) - M(1 + R_{fra} \tau) \times d(T^* - t) \]
Basis: definition

- **DEFINITION:** SPOT PRICE - FUTURES PRICE
  - $b_t = S_t - F_t$

- Depends on:
  - level of interest rate
  - time to maturity (↓ as maturity ↓)
Extension: Known cash income

\[ F_0 = (S_0 - I)e^{rT} \]

where \( I \) is the present value of the income

- Ex: forward contract to purchase a coupon-bearing bond

\[ r = 5\% \]

\[ S_0 = 110.76 \quad C = 6 \]

\[ I = 6 e^{-(0.05)(0.25)} = 5.85 \]

\[ F_0 = (110.76 - 5.85) e^{(0.05)(0.50)} = 107.57 \]
Known dividend yield

- \( q \): dividend yield p.a. paid continuously
  \[
  F_0 = \left[ e^{-qT} S_0 \right] e^{rT} = S_0 e^{(r-q)T}
  \]

- Examples:
- Forward contract on a Stock Index
  \( r \): interest rate
  \( q \): dividend yield
- Foreign exchange forward contract:
  \( r \): domestic interest rate (continuously compounded)
  \( q \): foreign interest rate (continuously compounded)
Interest rate parity

$r_s$: foreign interest rate (2%)

Underlying asset: one unit of foreign currency

$1$ → $\$e^{r_s T}$

$\$e^{2\% \times 0.5} = 1.0101$

$F_0$ forward exchange rate €/$

$\€F_0 e^{r_s T}$

$F_0 e^{r_s T} = S_0 e^{r_e T}$

$F_0 = S_0 e^{(r_e - r_s) T}$

Spot exchange rate €/$

$\€S_0$ → $\€S_0 e^{r_e T}$

$\€0.70 \times e^{4\% \times 0.50} = 0.7141$

Time 0 $r_e$: domestic interest rate (4%) Time $T$
Commodities

- \( I = - \text{PV of storage cost (negative income)} \)

- \( q = - \text{storage cost per annum as a proportion of commodity price} \)

The cost of carry:
- Interest costs + Storage cost – income earned \( c = r - q \)
- For consumption assets, short sales problematic. So:

\[
F_0 \leq S_0 e^{(r+u)T}
\]

- The \textit{convenience yield} on a consumption asset \( y \) defined so that:

\[
F_0 = S_0 e^{(c-y)T}
\]
## Summary

<table>
<thead>
<tr>
<th>Value of forward contract</th>
<th>Forward price</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No income</strong></td>
<td><strong>$F = S e^{rT}$</strong></td>
</tr>
<tr>
<td>$f = S - K e^{-rT}$</td>
<td><strong>$F = S e^{rT}$</strong></td>
</tr>
<tr>
<td><strong>Known income</strong></td>
<td><strong>$F = (S - I) e^{rT}$</strong></td>
</tr>
<tr>
<td>$I = PV(\text{Income})$</td>
<td></td>
</tr>
<tr>
<td>$f = (S - I) - K e^{-rT}$</td>
<td></td>
</tr>
<tr>
<td><strong>Known yield $q$</strong></td>
<td><strong>$F = S e^{(r-q)T}$</strong></td>
</tr>
<tr>
<td>$f = S e^{-qT} - K e^{-rT}$</td>
<td></td>
</tr>
<tr>
<td><strong>Commodities</strong></td>
<td><strong>$F = S e^{(r+u-y)T}$</strong></td>
</tr>
<tr>
<td>$f = S e^{(u-y)T} - K e^{-rT}$</td>
<td></td>
</tr>
</tbody>
</table>
Valuation of futures contracts

• If the interest rate is non stochastic, futures prices and forward prices are identical

• NOT INTUITIVELY OBVIOUS:
  – Total gain or loss equal for forward and futures
  – but timing is different
    • Forward: at maturity
    • Futures: daily
Forward price & expected future price

• Is $F_0$ an unbiased estimate of $E(S_T)$?

• $F < E(S_T)$ Normal backwardation

• $F > E(S_T)$ Contango

\[
F = E(S_T) e^{(r-k)T}
\]

$r =$ risk-free rate
$k =$ expected return
$= $ risk-free + risk premium

• If $k = r$ $F = E(S_T)$

• If $k > r$ $F < E(S_T)$

• If $k < r$ $F > E(S_T)$

$CAPM:$

\[
k = r + \beta (k_{Market} - r)
\]
Does the forward price predict the future price?

Consider a non dividend paying stock.

\[ S_0 = \frac{E_0(S_1)}{1 + E(R)} \iff E_0(S_1) = S_0(1 + E(R)) \]

Forward price:

\[ F_0 = S_0(1 + r_F) \]

\[ E_0(S_1) - F_0 = S_0 \left[ E(R) - r_F \right] \]

Forward price bias = Risk premium on underlying asset

This is also the expected cash flow on the derivative
Does the forward price predict the future price?

Binomial model illustration

Consider a non dividend paying stock.

Current stock price: 100
Expected return: 8%
Risk-free rate: 2%
Forward price: 102

\[ E(S_1 - F_0) = E(S_1) - F_0 = S_0(1 + k) - S_0(1 + r) = S_0(k - r) \]
Does the forward interest rate predict the future interest?

Binomial model illustration

Consider a 2-year zero coupon

<table>
<thead>
<tr>
<th>Current price</th>
<th>95,50</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-yr spot rate</td>
<td>2,30%</td>
</tr>
<tr>
<td>Expected return</td>
<td>2,93%</td>
</tr>
<tr>
<td>Risk Premium</td>
<td>0,93%</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>2,00%</td>
</tr>
<tr>
<td>Forward price</td>
<td>97,43</td>
</tr>
<tr>
<td>Forward int.rate</td>
<td>2,60%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1-yr rate</th>
<th>Proba</th>
<th>Return</th>
<th>CF Forward</th>
</tr>
</thead>
<tbody>
<tr>
<td>99,5</td>
<td>0,50%</td>
<td>60,00%</td>
<td>4,19%</td>
</tr>
<tr>
<td>95,50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>96,5</td>
<td>3,56%</td>
<td>40,00%</td>
<td>1,05%</td>
</tr>
</tbody>
</table>

Expected value | 98,3 |
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1,73%</td>
<td>2,93%</td>
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