Derivatives
Pricing a Forward / Futures Contract

Professor André Farber
Solvay Brussels School of Economics and Management
Université Libre de Bruxelles
Forward price and value of forward contract: review

• Forward price:

\[ F_0 = S_0 e^{rT} \]

• Remember: the forward price is the delivery price which sets the value of a forward contract equal to zero.

• Value of forward contract with delivery price \( K \)

\[ f = S_0 - Ke^{-rT} \]

• You can check that \( f = 0 \) for \( K = S_0 e^{rT} \)
Repurchase agreement (repo)

- An important kind of forward contract.
- Sale of a security + agreement to buy it back at a fixed price. (=reverse cash-and-carry, sale coupled with long forward)
- Underlying security held as a collateral
- Most repo are overnight
- Counterparty risk => haircut = Value of collateral - Loan
Schematic repo transaction

**Time $t$**

- **MARKET**
  - Buy bond at $Pt$
  - Pay $Pt$

- **TRADER**
  - Deliver bond
  - Get $Pt$ - haircut

**Time $T$**

- **MARKET**
  - Sell bond at $PT$
  - Get $PT$

- **TRADE**
  - Get bond

- **REPO DEALER**
  - Pay $(Pt - \text{Haircut})$
  - Pay $(1 + \text{RepoRate} \times (T-t))$
The repo market

Financial institution

Deposit (overnight)

Institutional investors
Non financial firms

Collateral
Market value + haircut

(No checking account insured by FDIC)

Huge repo market – exact size unknown
The banking panic

Increase in haircuts = withdrawal from banks

Fire sales

Source: Gorton and Metrick (2009a).
Spot rates and the yield curve

A yield curve is a representation of the relationship between market remuneration rates and the remaining time to maturity of debt securities, also known as the term structure of interest rates. The ECB publishes several yield curves as shown below.
Forward on a zero-coupon: example

- Consider a:
  - 6-month forward contract
  - on a 1-year zero-coupon with face value $A = 100$
  - Current interest rates (with continuous compounding)
    - 6-month spot rate: 4.00%
    - 1-year spot rate: 4.30%
- Step 1: Calculate current price of the 1-year zero-coupon
  - use 1-year spot rate
  $$S_0 = 100 e^{-(0.043)(1)} = 95.79$$
- Step 2: Forward price = future value of current price
  - use 6-month spot rate
  $$F_0 = 95.79 e^{(0.04)(0.50)} = 97.73$$
Forward on zero coupon

- Notations (continuous rates):
  \( A \equiv \) face value of ZC with maturity \( T^* \)
  \( r^* \equiv \) interest rate from 0 to \( T^* \)
  \( r \equiv \) interest rate from 0 to \( T \)
  \( R \equiv \) forward rate from \( T \) to \( T^* \)

- Value of underlying asset:
  \( S_0 = A e^{-r^* T^*} \)

- Forward price:
  \[ F_0 = S_0 e^{r T} = A e^{(r T - r^* T^*)} = A e^{-R(T^*-T)} \]
Forward interest rate

- Rate $R$ set at time 0 for a transaction (borrowing or lending) from $T$ to $T^*$
- With continuous compounding, $R$ is the solution of:
  \[ A = F_0 e^{R(T^* - T)} \]
  - The forward interest rate $R$ is the interest rate that you earn from $T$ to $T^*$ if you buy forward the zero-coupon with face value $A$ for a forward price $F_0$ to be paid at time $T$.
- Replacing $A$ and $F_0$ by their values:
  \[ R = \frac{r^* T^* - rT}{T^* - T} \]

In previous example:
  \[ R = \frac{4.30\% \times 1 - 4\% \times 0.50}{1 - 0.50} = 4.60\% \]
Forward rate: Example

- Current term structure of interest rates (continuous):
  - 6 months: 4.00% \( \Rightarrow \) \( d = \exp(-0.040 \times 0.50) = 0.9802 \)
  - 12 months: 4.30% \( \Rightarrow \) \( d^* = \exp(-0.043 \times 1) = 0.9579 \)
- Consider a 12-month zero coupon with \( A = 100 \)
- The spot price is \( S_0 = 100 \times 0.9579 = 95.79 \)
- The forward price for a 6-month contract would be:
  - \( F_0 = 97.79 \times \exp(0.04 \times 0.50) = 97.73 \)
- The continuous forward rate is the solution of:
  - \( 97.73 \ e^{0.50R} = 100 \)
  \( \Rightarrow R = 4.60\% \)
Forward rate - illustration

Kamakura Corporation
10 Year Forecast of US Treasury Yield Curve Implied by Forward Rates
Using Maximum Smoothness Forward Rate Smoothing

U.S. Treasury Maturity

Months Forward
For forward borrowing:

- View forward borrowing as a forward contract on a ZC.
  - You plan to borrow $M$ for $\tau$ years from $T$ to $T^*$
  - Define: $\tau = T^* - T$
  - The simple interest rate set today is $R_S$
  - You will repay $M(1 + R_S \tau)$ at maturity

- In fact, you sell forward a ZC.
  - The face value is $M(1 + R_S \tau)$
  - The maturity is $T^*$
  - The delivery price set today is $M$

- The interest will set the value of this contract to zero.
Forward borrowing: Gain/loss

- At time $T^*$:
- Difference between the interest paid $R_S$ and the interest on a loan made at the spot interest rate at time $T$ : $r_s$
  $$M [ r_s - R_s \tau ]$$

- At time $T$:
  - $\Pi_T = [M ( r_s - R_s \tau )] / (1+r_s\tau)$
FRA (Forward rate agreement)

- **Example: 3/9 FRA**
  - *Buyer* pays fixed interest rate \( R_{fra} \) 5%
  - *Seller* pays variable interest rate \( r_s \) 6-m LIBOR
  - on notional amount \( M \) $100 m
  - for a given time period (contract period) \( \tau \) 6 months
  - at a future date (settlement date or reference date) \( T \) (in 3 months, end of accrual period)

- Cash flow for buyer (long) at time \( T \):
  - Inflow \((100 \times \text{LIBOR} \times 6/12)/(1 + \text{LIBOR} \times 6/12)\)
  - Outflow \((100 \times 5\% \times 6/12)/(1 + \text{LIBOR} \times 6/12)\)

- Cash settlement of the difference between present values
FRA: Cash flows 3/9 FRA (buyer)

- General formula: 
  \[ CF = M[(r_S - R_{fra})\tau]/(1+r_S \tau) \]

- Same as for forward borrowing - long FRA equivalent to cash settlement of result on forward borrowing

\[
(100 \times 5\% \times 0.50)/(1+r_S 0.50)
\]
**Review: Long forward (futures)**

Synthetic long forward: borrow $S_0$ and buy spot

$$f_0 = 0$$

$$F_0 = S_0 e^{rT}$$

$$f_t = S_t - F_0 e^{-rT} = (F_t - F_0) e^{-rT}$$

Receive underlying asset

Pay forward price

Value of forward contract
Review Long forward on zero-coupon

1. Face value of zero-coupon given

\[ S_0 = Ae^{-rT} \]

2. Calculate forward price which sets the initial value of the contract equal to 0.

\[ F_0 = S_0e^{rT} \]

3. Calculate forward interest rate

\[ A = F_0e^{\tau} \]

\[ f_t = S_t - F_0e^{-rT} = (F_t - F_0)e^{-rT} \]
Review Forward investment = buy forward zero-coupon

\[ S_0 = Ae^{-r^* T^*} \]

\[ f_0 = 0 \]

\[ S_0 \]

\[ f_T = S_T - M = \frac{M (1 + R_S \tau)}{1 + r_s \tau} - M = \frac{M (R_S - r_s) \tau}{1 + r_s \tau} \]

(1) Forward price given

(2) Calculate face value of zero-coupon which set the value of the contract equal to 0

\[ A = M (1 + R_S \tau) = F_0 e^{R \tau} \]
Review Forward borrowing=sell forward zero-coupon

\[ S_0 = Ae^{-r*TT^*} \]

\[ f_0 = 0 \]

\[ S_0 = \frac{M(1 + R_S\tau)}{1 + r_s\tau} = \frac{M(r_s - R_S)\tau}{1 + r_s\tau} \]

\[ F_0 = M \]

\[ A = M(1 + R_S\tau) = F_0 e^{R\tau} \]
Review Long Forward Rate Agreement

\[ S_0 = Ae^{-r^* \tau} \]

\[ f_0 = 0 \]

\[ \frac{M r_s \tau}{1 + r_s \tau} \]

\[ \frac{M R_{fra} \tau}{1 + r_s \tau} \]

\[ f_T = \frac{M (r_s - R_{fra}) \tau}{1 + r_s \tau} \]
To lock in future interest rate

Borrow short & invest long

Buy forward zero coupon

Invest forward at current forward interest rate

Sell FRA and invest at future (unknown) spot interest rate
Valuing a FRA

\[
\frac{Mr_s \tau}{1 + r_s \tau} + \frac{M}{1 + r_s \tau} = M
\]

\[
f_t = M \times d(T - t) - M(1 + R_{fra} \tau) \times d(T^* - t)
\]
Basis: definition

- DEFINITION: SPOT PRICE - FUTURES PRICE
  - \( b_t = S_t - F_t \)

- Depends on:
  - level of interest rate
  - time to maturity (\( \downarrow \) as maturity \( \downarrow \))
Extension: Known cash income

\[ F_0 = (S_0 - I) e^{rT} \]

where \( I \) is the present value of the income

- Ex: forward contract to purchase a coupon-bearing bond

\[ r = 5\% \quad 0 \quad 0.25 \quad T = 0.50 \]

\[ S_0 = 110.76 \quad C = 6 \]

\[ I = 6 \ e^{-(0.05)(0.25)} = 5.85 \]

\[ F_0 = (110.76 - 5.85) \ e^{(0.05)(0.50)} = 107.57 \]
Known dividend yield

- $q$ : dividend yield p.a. paid continuously
  
  $$F_0 = \left[ e^{-qT} S_0 \right] e^{rT} = S_0 e^{(r-q)T}$$

- Examples:
- Forward contract on a Stock Index
  - $r$ = interest rate
  - $q$ = dividend yield
- Foreign exchange forward contract:
  - $r$ = domestic interest rate (continuously compounded)
  - $q$ = foreign interest rate (continuously compounded)
Interest rate parity

$r_s$: foreign interest rate (2%)

$\$1 \rightarrow \$e^{r_sT}$

$e^{2\% \times 0.5} = 1.0101$

$F_0$ forward exchange rate €/$

$\€F_0e^{r_sT}$

$F_0e^{r_sT} = S_0e^{r_eT}$

$F_0 = S_0e^{(r_e-r_s)T}$

Spot exchange rate €/$

$\€S_0 \rightarrow \€S_0e^{r_eT}$

$\€0.70 \times e^{4\% \times 0.50} = 0.7141$

Time 0 $r_e$: domestic interest rate (4%) Time $T$

Underlying asset: one unit of foreign currency
Commodities

- $I = - \text{PV of storage cost (negative income)}$

- $q = - \text{storage cost per annum as a proportion of commodity price}$

- The cost of carry:
  - Interest costs + Storage cost – income earned $c = r - q$
  - For consumption assets, short sales problematic. So:
    \[ F_0 \leq S_0 e^{(r+u)T} \]

- The *convenience yield* on a consumption asset $y$ defined so that:
  \[ F_0 = S_0 e^{(c - y)T} \]
**Summary**

<table>
<thead>
<tr>
<th></th>
<th>Value of forward contract</th>
<th>Forward price</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No income</strong></td>
<td>( f = S - Ke^{-rT} )</td>
<td>( F = Se^{rT} )</td>
</tr>
<tr>
<td><strong>Known income</strong></td>
<td>( f = (S - I) - Ke^{-rT} )</td>
<td>( F = (S - I)e^{rT} )</td>
</tr>
<tr>
<td><strong>Known yield</strong></td>
<td>( f = Se^{-qT} - Ke^{-rT} )</td>
<td>( F = Se^{(r-q)T} )</td>
</tr>
<tr>
<td><strong>Commodities</strong></td>
<td>( f = Se^{(u-y)T} - Ke^{-rT} )</td>
<td>( F = Se^{(r+u-y)T} )</td>
</tr>
</tbody>
</table>
Valuation of futures contracts

- If the interest rate is non stochastic, futures prices and forward prices are identical
- NOT INTUITIVELY OBVIOUS:
  - Total gain or loss equal for forward and futures
  - but timing is different
    - Forward: at maturity
    - Futures: daily
Forward price & expected future price

- Is $F_0$ an unbiased estimate of $E(S_T)$?
- $F < E(S_T)$  Normal backwardation
- $F > E(S_T)$  Contango

- $F = E(S_T) \, e^{(r-k)\,(T-t)}$
- If $k = r$  $F = E(S_T)$
- If $k > r$  $F < E(S_T)$
- If $k < r$  $F > E(S_T)$
Does the forward price predict the future price?

Binomial model illustration
Consider a non dividend paying stock.

Current stock price 100
Expected return 8%
Risk-free rate 2%
Forward price 102
Strike price 102

\[
\begin{array}{c|c|c|c}
& \text{Proba} & \text{Return} & \text{CF Forward} \\
100 & 62.22\% & 25.0\% & 23 \\
80 & 37.78\% & -20.0\% & -22 \\
\end{array}
\]

Expected value 108 8.0\% 6