Random Walks in the Sky
(remembering your steps!)

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(arXiv: 1201.3876, 1205.3401, 1305.0724, 1306.0551)

October 2013
Formation of structures

Halos evolve from critical overdensities

Can use halos to reconstruct the primordial distribution
Formation of structures

Early Universe (Inflation?)

Small density inhomogeneities ($\sim 10^{-5}$)

Expansion, Quantum fluctuations

Linearized GR

Galaxies, halos, voids, filaments...

Structure formation

Non-linear Newtonian gravity

Question mark
Halo Mass Function and Bias

- How many halos of given mass (mass function)
- Correlation of halo counts with underlying matter field (halo bias)

GOALS:

- Analytical predictions for halo statistics reproducing N-body sims
- Information on initial conditions and matter/energy balance from data:

  Properties of Early Universe
  Energy and matter content
  $\Lambda$ vs Quintessence vs ModGrav
Part ONE

- The excursion set approach
- Analytical progress for mass functions: the UPWARDS approximation

Part TWO

- Bias from excursion sets (a simple way!)
Halos from “dense enough” patches in the initial matter distribution $\delta_{in}$

Mean density $\delta_R \equiv \left\langle \delta_{in} \right\rangle \geq \text{threshold } B$

$$\delta_R(x) \equiv \frac{1}{V_R} \int d^3y \ W_R(y - x) \delta(y) \geq B$$

Halo mass $M$ proportional to the volume $V \sim R^3$ of the patch.

$$M = \bar{\rho} \ V_R \equiv \bar{\rho} \int d^3y \ W_R(y)$$

If $B$ from spherical collapse in Einstein-DeSitter, then $B = \delta_c = 1.686$. But in general $B$ is sensitive to $\Omega_m$, $\Omega_\Lambda$, $w$, baryons, neutrinos, modifications of gravity... It depends on scale and redshift

Correlations depend on matter power spectrum $P(k)$ and choice of $W$
Different locations realize different random walks: \( s(M) \equiv \langle \delta_R^2(x) \rangle \)

- FIRST PASSAGE PROBLEM!
- CORRELATED STEPS!
  - No known solution
  - NEED BETTER MATHS

Abundance \( n(M) \leftrightarrow \) first crossing probability \( f(s) \) at scale \( s(M) \)

Oversimplified picture, working surprisingly well
Probability of ANY crossing at $s$:

$$f(s) = \frac{d}{ds} \langle \vartheta(\delta - b(s)) \rangle$$

Press & Schechter (1974)

Not any, but **FIRST** crossing (cloud-in-cloud problem); solution only for Gaussian uncorrelated steps

$$f(s) = \frac{\langle \vartheta(b_1 - \delta_1) \cdots \vartheta(b_{N-1} - \delta_{N-1}) \vartheta(\delta_N - b_N) \rangle}{\Delta s}$$

Bond et al. (1991)
First crossing distribution

- However: strongly correlated walks are less affected (less zig-zags)

  Paranjape, Lam & Sheth (2011)

- Can relax FIRST into simply **UPWARDS**: \( \delta = B ; \delta' \geq B' \)

\[
f(s) = \langle \left[ \frac{d}{ds} \varphi(\delta_s - B) \right] \varphi(\delta'_s - B') \rangle = \int_{B'}^{\infty} dv \ (v - B') \ p(B, v; s)
\]

MM & Sheth (2012)

\[
f(s) = -\left( \frac{B}{\sqrt{s}} \right)' e^{-B^2/2s} \left[ \frac{1 + \text{erf}(X/\sqrt{2})}{2} + \frac{e^{-X^2/2}}{2X \sqrt{2\pi}} \right]
\]
First crossing distribution

\[ \ln(10) y f(y) \]

\[ \lg(y = \frac{\delta^2}{\sigma^2}) \]

\( \alpha = -1 \)

\( \alpha = 0 \)

\( \alpha = 0.5 \)

\( \Lambda CDM \)

inverse gaussian

MM & Sheth (2012)
To reach any $\delta > b$ at $s$, must cross at some $S < s$:

$$p(\delta \geq b, s) = \int_0^S dS f(S)p(\delta \geq b, s|\text{first}, S)$$

Do the **UPWARDS** approximation $\delta'(S) \geq b'(S)$ and solve for $f(S)$

$$p(\delta \geq b, s|\text{first}, S) \quad \Rightarrow \quad p(\delta \geq b, s|\text{up}, S)$$

Now solve for $f(S)$

MM & Sheth (2013)
Upward mobility, back-substitution

\[ p(\delta \geq b, s) = \int_0^s dS f(S) p(\delta \geq b, s|\text{up, } S) \]

MM & Sheth (2013)
Upward mobility, back-substitution

$$p(\delta \geq b, s) = \int_0^s dS f(S)p(\delta \geq b, s|_{\text{up, } S})$$

- Full understanding of the correlated random walk problem, with any power spectrum and barrier!

- Does this $f(s)$ reproduce N-body mass function? Not really...

- Not surprising, space correlations are neglected. HOWEVER...

MM & Sheth (2013)
Upward mobility, back-substitution

$$p(\delta \geq b, s) = \int_0^s dS f(S) p(\delta \geq b, s \mid \text{up, } S)$$

HOWEVER:

- Simple model with Markovian velocities (not heights!)
- Rescaled constant barrier:

  $$\delta_c \rightarrow \sqrt{0.7}\delta_c$$

IT WORKS!!
Same formalism for non-Gaussian initial conditions:

\[
f(s) = \left[ \frac{d}{ds} \int_{b(s)}^{\infty} d\delta \, p(\delta; s) \right] \left( 1 + \frac{\text{erf}(X/\sqrt{2})}{2} \right) - B'p(B; s) \left[ \frac{e^{-X^2/2}}{2X \sqrt{2\pi}} + \ldots \right]
\]

- Non-perturbative in NG parameters: \( p(\delta; s) \) is the exact pdf!

\[
p(B; s) = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{B^2}{2} + \frac{\mu B^3}{3!} + \ldots \right]
\]

\[
\mu = \frac{\langle \delta^3 \rangle}{s^{3/2}} \sim f_{\text{NL}}
\]

- Residual NG corrections are small: OK as perturbations

MM & Sheth (2013)
Light vs Mass: Halo Bias

- Relation between halo abundance $\delta_h$ and underlying DM density

- Usually, computed from $p(\delta_h; s | \delta_0; s_0)$. Is there an easier way?

- Yes, there is. Expanding halo correlation functions in terms of DM correlation functions
Take the most generic dependence on the matter field:

\[
\delta_h(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \int d^3y_1 \ldots d^3y_k \ b_k(x - y_1, \ldots, x - y_k) \delta(y_1) \ldots \delta(y_k)
\]

e.g. Matsubara (2011)

Compute connected correlation function of \( \delta_h \) and \( \delta \):

\[
\langle \delta_h(x) \delta(z_1) \ldots \delta(z_n) \rangle_c
\]

\[
= \int d^3x_1 \ldots d^3x_n \left\langle \frac{\delta^n \delta_h(x)}{\delta \delta(x_1) \ldots \delta \delta(x_n)} \right\rangle \prod_{j=1}^{n} \langle \delta(x_j) \delta(z_j) \rangle
\]

Bias functions

\( \delta_h \) acts as an effective vertex for \( \delta \)
Halo Bias from Excursion Sets

- Need a prediction for \( \delta_h \). Can get it from excursion sets:

\[
1 + \delta_h(m) = \frac{\vartheta(B_1 - \delta_1) \ldots \vartheta(B_{N-1} - \delta_{N-1}) \vartheta(\delta_N - B_N)}{\langle \vartheta(B_1 - \delta_1) \ldots \vartheta(B_{N-1} - \delta_{N-1}) \vartheta(\delta_N - B_N) \rangle}
\]

\[
b_n(x - y_1, \ldots, x - y_n) = \sum_{i_1, \ldots, i_n}^{N} \left\langle \frac{\partial^n \delta_h(m)}{\partial \delta_{i_1} \ldots \partial \delta_{i_n}} \right\rangle W_{i_1}(x - y_1) \ldots W_{i_n}(x - y_n)
\]

\[
\left\langle \frac{\partial^n \delta_h}{\partial \delta_{i_1} \ldots \partial \delta_{i_n}} \right\rangle = \frac{(-1)^n}{f(s)} \frac{\partial^n f(s)}{\partial B_{i_1} \ldots \partial B_{i_n}}
\]

- Still a bit complicated...

\[
\langle \delta_h \delta_0 \rangle = \sum_{i=1}^{N} \left\langle \frac{\partial \delta_h}{\partial \delta_i} \right\rangle \langle \delta_i \delta_0 \rangle \quad \langle \delta_h \delta_0^2 \rangle = \sum_{i, j=1}^{N} \left\langle \frac{\partial^2 \delta_h}{\partial \delta_i \partial \delta_j} \right\rangle \langle \delta_i \delta_0 \rangle \langle \delta_j \delta_0 \rangle
\]
Use the **UPWARDS** approximation. Only two variables!

\[
1 + \delta_h = \frac{1}{f(s)} \left[ \frac{d}{ds} \vartheta(\delta_s - B) \right] \vartheta(\delta' - B')
\]

The real space bias functions become easy:

\[
b_1(x - y) = -\frac{1}{f(s)} \left[ W_R(x - y) \frac{\partial}{\partial B} + W'_R(x - y) \frac{\partial}{\partial B'} \right] f(s)
\]

\[
b_n(x - y_1, \ldots, x - y_n) = \frac{(-1)^n}{f(s)} \prod_{i=1}^{n} \left[ W_R(x - y_i) \frac{\partial}{\partial B} + W'_R(x - y_i) \frac{\partial}{\partial B'} \right] f(s)
\]
Halo Bias from Excursion Sets

- What should one measure?

\[
\langle \delta_h \delta_0 \rangle = b_{10}^{(f)} \langle \delta \delta_0 \rangle + b_{11}^{(f)} \langle \delta' \delta_0 \rangle
\]

\[
\langle \delta_h \delta_0^2 \rangle_c = b_{20}^{(f)} \langle \delta \delta_0 \rangle^2 + 2b_{21}^{(f)} \langle \delta \delta_0 \rangle \langle \delta' \delta_0 \rangle + b_{22}^{(f)} \langle \delta' \delta_0 \rangle^2
\]

MM, Paranjape & Sheth (2012)

- The coefficients are straightforward:

\[
b_{nk}^{(f)} = \frac{(-1)^n}{f(s)} \frac{\partial^{n-k}}{\partial B^{n-k}} \frac{\partial^k}{\partial B'_{nk}} f(s)
\]

with

\[
f(s) = \int_{B'}^{\infty} dv (v - B') p(B, v)
\]

MM, Paranjape & Sheth (to appear)
Halo Bias from Excursion Sets

- Linear bias in Fourier space:
  \[ b_1(k) = b^{(f)}_{10} W(kR) + b^{(f)}_{11} 2sW'(kR) \]
  - \( b^{(f)}_{10} \sim 1 \)
  - \( b^{(f)}_{11} \sim k^2 R^2 \)

- Quadratic bias:
  \[ b_2(k_1, k_2) \simeq b^{(f)}_{20} + b^{(f)}_{21} (k_1^2 + k_2^2) R^2 + b^{(f)}_{22} k_1^2 k_2^2 R^4 \]

- Numerical predictions for the coefficients \( b_{n,j}(m) \) from \( f(s) \); \( b_{n0} \) is the same as in peak-background split

- \( k \)-dependence!

MM, Paranjape & Sheth (2012)
Adding non-Gaussianity: Bias

- Expand halo-matter $n$-point functions in matter polyspectra

\[
\langle \delta_h(x)\delta(z_1) \rangle_c = \bullet + \circ \quad + \ldots
\]

\[
\langle \delta_h(x)\delta(z_1)\delta(z_2) \rangle_c = \circ \bullet + \bullet \circ \quad + \ldots
\]

- Generic excursion set bias for non-Gaussian walks:

\[
\langle \delta_h \delta_0 \rangle = \sum_{i=1}^{N} \left( \frac{\partial \delta_h}{\partial \delta_i} \right) \langle \delta_i \delta_0 \rangle + \frac{1}{2} \sum_{i,j=1}^{N} \left( \frac{\partial^2 \delta_h}{\partial \delta_i \partial \delta_j} \right) \langle \delta_i \delta_j \delta_0 \rangle + \ldots
\]
Adding non-Gaussianity: Bias

With the two-step approximation:

\[
\langle \delta h \delta_0 \rangle \simeq b_{10}^{(f)} \langle \delta \delta_0 \rangle + b_{11}^{(f)} \langle \delta' \delta_0 \rangle \\
+ \frac{1}{2} \left[ b_{20}^{(f)} \langle \delta^2 \delta_0 \rangle + 2 b_{21}^{(f)} \langle \delta' \delta \delta_0 \rangle + b_{22}^{(f)} \langle \delta'^2 \delta_0 \rangle \right] + \ldots
\]

Same definition of \( b_{nj} \) as before, but now with non-Gaussian \( f(s) \)

\[
\Delta b_1(k) = \frac{2 f_{\text{NL}}^{\text{local}}}{k^2 T(k)} \left[ s b_{20}^{(f)} + b_{21}^{(f)} + \langle (\delta')^2 \rangle b_{22}^{(f)} + \mathcal{O}(k^2) \right]
\]

All coefficients matter at small \( k \). Possibly, also some effects at \( k \sim R \) (where the leading term starts decaying). Equilateral NG?

MM, Paranjape & Sheth (to appear)
Conclusions

- Accurate solution of first passage of correlated random walks
- Full understanding of the excursion set approach to structure formation
- Simple rescaling of the spherical collapse barrier reproduces correctly the Gaussian mass function
- Straightforward non-perturbative inclusion of NG (can do Eulerian field!)
- Self-consistent predictions of bias functions and coefficients, new strategies to measure them in simulations
- To do: check against N-body simulations, generalization to excursion set theory of peaks
- Interesting possibilities (?) for primordial non-Gaussianity

Thanks!!