Lectures on Kac-Moody Algebras with Applications in (Super-)Gravity

Daniel Persson and Nassiba Tabti

Physique Théorique et Mathématique,
Université Libre de Bruxelles & International Solvay Institutes,
ULB-Campus Plaine C.P. 231, B-1050, Bruxelles, Belgium

e-mail: `dpersson, ntabti@ulb.ac.be`

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Abstract

These lectures are divided into two main parts. In the first part we give an introduction to the theory of Kac-Moody algebras directed towards physicists. In particular, we describe the subclasses of affine and Lorentzian Kac-Moody algebras in detail. Our treatment focuses on the Chevalley-Serre presentation, and emphasizes the importance of the Weyl group. We illustrate the basic theory with simple examples. In the second part of the lectures we make use of the contents of part one to describe some recent developments devoted to investigations of the underlying symmetry structures of supergravity theories. We begin by describing how toroidal compactifications of gravity theories reveals hidden global and local symmetries of the reduced Lagrangian. We also discuss attempts at extending these symmetry structures to infinite-dimensional algebras. We show how a manifestly Kac-Moody invariant action can be constructed, whose solutions correspond to exact BPS solutions in maximal supergravity theories.

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Chapter 1

Introduction

The theory of finite-dimensional (semi-)simple Lie algebras has played a major role in the development of theoretical high energy physics as we know it today. Moreover, with the advent of string theory physicists were naturally led to the study of infinite-dimensional generalizations of Lie algebras, known as affine Kac-Moody algebras. These describe the algebraic structure of the holomorphic and anti-holomorphic currents of the two-dimensional conformal field theory living on the worldsheet of the string. Long ago it was speculated that since affine algebras are in this way closely linked to the quantization of a single string, further extensions, such as hyperbolic Kac-Moody algebras, should be intimately connected with the quantization of a multitude of strings, i.e., to string field theory [1]. Unfortunately, no concrete realization of this intriguing idea has been found as of yet.

However, it turns out that Lorentzian Kac-Moody algebras have made their way to the stage from a slightly different path. At the end of the seventies Cremmer and Julia found that the toroidal compactification of eleven-dimensional supergravity to $11-n$ dimensions reveals a chain of global hidden symmetries described by the Lie groups $\mathcal{E}_{n(n)}$, culminating with the largest exceptional simple Lie group $\mathcal{E}_{8(8)}$ in three-dimensions [2, 3]. Reduction to dimensions below three is more complicated however due to the fact that the symmetry groups are expected to become infinite-dimensional, with the affine symmetry group $\mathcal{E}_{9(9)}$ appearing in two dimensions, the hyperbolic group $\mathcal{E}_{10(10)}$ in one dimension and perhaps even the Lorentzian group $\mathcal{E}_{11(11)}$ in the reduction to zero dimensions. Although this is highly speculative, it was conjectured by Julia that all the symmetry groups $\mathcal{E}_{n(n)}$ should, in fact, be present for $n \leq 11$ [4].

Subsequently, part of this conjecture was verified through the discovery of a global $\mathcal{E}_{9(9)}$-symmetry of the space of solutions of $\mathcal{N} = 16$ supergravity in two dimensions [5]. This symmetry can be seen as a generalization of the affine symmetry group $SL(2,\mathbb{R})^+$, known as the "Geroch group", appearing in the reduction of pure four-dimensional gravity to two dimensions [6, 7]. In this way, affine Kac-Moody algebras have become relevant also in classical theories, and not only within the quantum realm as described above.

But what about the more mysterious algebraic structures described by Lorentzian Kac-Moody algebras? Do they also play some fundamental roles as (hidden) underlying symmetries of supergravity and string/M-theory? A pioneering step aimed at answering these questions was taken by West in [8], where he managed to reformulate eleven-dimensional supergravity as a non-linear realization based on (a "truncated" version of) the Lorentzian Kac-Moody algebra $E_{11(11)} = \mathrm{Lie}[\mathcal{E}_{11(11)}]$. This led him to conjecture that the full group $\mathcal{E}_{11(11)}$ should be a symmetry of M-theory itself (see [9] for a recent review).

The study of gravitational theories close to a spacelike singularity has provided further evidence for the existence of infinite algebraic symmetry structures of gravity. It has been found that in this limit (the "BKL-limit") the dynamics of eleven-dimensional supergravity at each spatial point is controlled by the Weyl group of the hyperbolic Kac-Moody algebra $E_{10(10)} = \mathrm{Lie}[\mathcal{E}_{10(10)}]$ [10]. This discovery led Damour, Henneaux and Nicolai to conjecture

that the full group $\mathcal{E}_{10(10)}$ should be realized as a symmetry of M-theory [11] (see [12] and [13] for reviews). This conjecture was partially verified for a suitably truncated version of the theory. A proposal for a unified treatment of this conjecture and that of West was put forward in [14].

The plan of these lecture notes is as follows. In Chapter 2 we develop the preliminary mathematical background material required to understand subsequent chapters. We explain how to construct Kac-Moody algebras from Cartan matrices using the Chevalley-Serre presentation. We introduce the important concept of the Weyl group and explain how it can be used to describe the root systems of Kac-Moody algebras. We focus our attention on the affine and Lorentzian Kac-Moody algebras, where for the latter class we discuss in particular the hyperbolic subset. Throughout the text we illustrate the abstract theory with explicit examples based on the finite simple Lie algebras A_1 and A_2, as well as the affine and hyperbolic Kac-Moody algebras A_1^+ and A_1^{++}, respectively.

In Chapter 3 we explain how hidden Kac-Moody symmetries are exhibited through compactifications of gravitational theories on a torus. First, we recall briefly the Kaluza-Klein reduction of pure gravity on S^1. Then, we consider in detail the reduction of gravity on T^2. In these simple examples we introduce the essential aspects of enhanced symmetries, most notably that of scalar coset Lagrangians and non-linear realisations. After this, we study the reduction of pure gravity and 11-dimensional supergravity to 3 dimensions. This leads to the enhancement of the symmetries to a certain finite Lie group $\mathcal{G}$. Finally, we explain how infinite dimensional Kac-Moody symmetries $\mathcal{G}^{+++}$ characterise pure gravitational theories and eleven-dimensional supergravity through reduction below three dimensions.

In Chapter 4, we construct an action explicitly invariant under $\mathcal{G}^{+++}$ to make the Kac-Moody symmetries manifest. To this end we introduce the notion of level decomposition and temporal involution. We study the $\mathcal{G}^{++}$-invariant actions: $\mathcal{G}_C^{++}$ (the "brane action") and $\mathcal{G}_B^{++}$ (the "cosmological action") to make connections between this new formalism and the covariant spacetime theories. We also review some important consequences of Weyl transformations. Finally, we give conclusions and future perspectives in Chapter 5.

Chapter 2

Kac-Moody Algebras

In this chapter we present the basic theory of Kac-Moody algebras, with emphasis on the affine and Lorentzian cases. Our treatment is aimed towards physicists, and to this end we do not give formal definitions, theorems or proofs, but rather we introduce the reader to a "toolbox", whose constituents can, in principle, be mastered relatively quickly. We presuppose a working knowledge of the theory of finite simple Lie algebras, and explain in detail how these structures generalize to arbitrary infinite-dimensional Kac-Moody algebras. Recommended references for this chapter are [15, 16]. For a treatment in the same spirit as the present one, with partial overlap, see [13].

2.1 Preliminary Example: A_1 – "The Mother of all Lie Algebras"

In this section we consider a simple "warm-up" example which nevertheless contains many of the important features encountered later on. Even though being the smallest finite-dimensional simple Lie algebra, A_1 plays an important role in the general theory of Kac-Moody algebras. In particular, one may view any simple rank r Kac-Moody algebra $\mathfrak{g}$ as a set of r distinct A_1-subalgebras which are intertwined in a non-trivial way. This fact is a cornerstone in the representation theory of Kac-Moody algebras [15]. For this reason, A_1 has been called "the Mother of all Lie algebras" [17]. All terminology introduced in this example will be properly defined in subsequent sections.

$A_1 \simeq \mathfrak{sl}(2, \mathbb{C})$ is the algebra of 2×2 complex traceless matrices. This algebra is 3-dimensional and we take as a basis the set of generators $\{T_i \mid i = 1, 2, 3\}$, subject to the commutation relations

$$[T_i, T_j] = -\varepsilon_{ijk} T_k, \tag{2.1.1}$$

with $\varepsilon_{123} = 1$. Writing this out we find the following relations between the generators:

$$[T_1, T_2] = -T_3, \qquad [T_1, T_3] = T_2, \qquad [T_2, T_3] = -T_1. \tag{2.1.2}$$

In the fundamental representation we have a matrix realization of the form

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad T_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{2.1.3}$$

which can easily be seen to satisfy (2.1.2). For the purposes of this paper, it is useful to

switch to another basis, where the three basis elements take the form

$$\begin{aligned} e &\equiv & T_2 - iT_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ f &\equiv & -(T_2 + iT_1) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ h &\equiv & -2iT_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.1.4}$$

The new basis $\{e, f, h\}$ is similar to the familiar basis $\{J^+, J^-, J^3\}$ of $\mathfrak{su}(2)$. The commutation relations now become

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f, \tag{2.1.5}$$

implying that the generators e and f can be thought of a "step operators", taking us, respectively, "up" and "down" between the lowest and highest weights of the representation. The basis $\{e, f, h\}$ is called the Chevalley basis, and (2.1.5) corresponds to a Chevalley presentation of A_1. There are two main reasons for working in this basis. Firstly, it is the starting point for the generalization to arbitrary Kac-Moody algebras which we shall consider in the next section. Secondly, in the Chevalley basis, the matrix realization of the generators only involves *real* traceless matrices. This ensures that simply by restricting all linear combinations of generators to the real numbers, we obtain a *real* Lie algebra, namely the split real form $\mathfrak{sl}(2, \mathbb{R})$, consisting of 2×2 real traceless matrices.

The Chevalley presentation reveals a natural decomposition of $\mathfrak{sl}(2, \mathbb{R})$ as the following direct sum of vector spaces

$$\mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e. \tag{2.1.6}$$

This is the triangular decomposition of the Lie algebra, which in this case simply means that each matrix can be decomposed as a sum of a lower triangular, a diagonal and an upper triangular matrix. The two subspaces $\mathbb{R}f$ and $\mathbb{R}e$ form nilpotent subalgebras and $\mathbb{R}h$ form the abelian Cartan subalgebra. Of course, these concepts are all somewhat trivial in this example, but it is nevertheless useful to employ this terminology for later use.

An important concept which we shall encounter numerous times is that of a maximal compact subalgebra. It is well known that the maximal compact subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ is $\mathfrak{so}(2)$, the algebra of 2×2 (traceless) antisymmetric matrices. Let us now try to understand this from a more abstract point of view. The algebra $\mathfrak{so}(2) \subset \mathfrak{sl}(2, \mathbb{R})$ is a so-called involutory subalgebra, meaning that there exists an involution ω of $\mathfrak{sl}(2, \mathbb{R})$ such that $\mathfrak{so}(2)$ coincides with the set $\{x \in \mathfrak{sl}(2, \mathbb{R}) \mid \omega(x) = x\}$, which is pointwise fixed by ω. The involution ω is called the Chevalley involution and is defined on the Chevalley generators as follows

$$\omega(e) = -f, \qquad \omega(f) = -e, \qquad \omega(h) = -h. \tag{2.1.7}$$

This is obviously an involution, $\omega^2 = 1$, and it leaves the commutation relations invariant, so ω is an automorphism of $\mathfrak{sl}(2, \mathbb{R})$.

Now note that the combination $e - f$ is fixed by ω and so spans the one-dimensional maximal compact subalgebra of $\mathfrak{sl}(2, \mathbb{R})$. Indeed we have

$$e - f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{so}(2), \tag{2.1.8}$$

so that we may write $\mathfrak{so}(2) = \mathbb{R}(e - f)$. The Chevalley involution thus induces another decomposition of $\mathfrak{sl}(2, \mathbb{R})$ into pointwise invariant and anti-invariant subsets under the action of ω. Explicitly, this yields the Cartan decomposition

$$\mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}(e - f) \oplus \Big(\mathbb{R}h \oplus \mathbb{R}(e + f)\Big). \tag{2.1.9}$$

It is important to note that this is a direct sum of vector spaces, and, in particular, that the anti-invariant part $\mathfrak{p} \equiv \mathbb{R}h \oplus \mathbb{R}(e+f)$ is *not* a subalgebra. This is in contrast to the triangular decomposition above, for which each subspace is a subalgebra in itself. It is easy to see that $\mathfrak{p}$ does not form a subalgebra by noticing that the commutation relations do not close,

$$[h, e+f] = 2(e-f) \in \mathfrak{so}(2). \tag{2.1.10}$$

Moreover we have

$$[h, e-f] = 2(e+f) \in \mathfrak{p}, \qquad [e-f, e+f] = 2h \in \mathfrak{p}, \tag{2.1.11}$$

revealing that the algebraic structure of the decomposition is

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{so}(2), \qquad [\mathfrak{so}(2), \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{so}(2), \mathfrak{so}(2)] \subset \mathfrak{so}(2), \tag{2.1.12}$$

indicating that at the "group level" $\mathfrak{p}$ corresponds to a symmetric space, which in this case coincides with the coset space $SL(2,\mathbb{R})/SO(2)$. We shall come back to this in Section 2.2.3.

Finally, we consider an additional useful decomposition, known as the Iwasawa decomposition, which will play an important role in what follows. It takes the form (direct sum of vector spaces)

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2) \oplus \mathbb{R}h \oplus \mathbb{R}e, \tag{2.1.13}$$

where each subspace is now a subalgebra. Note that in this decomposition, even though $\mathfrak{so}(2)$ is defined as before through the Chevalley involution, the second part, $\mathbb{R}h \oplus \mathbb{R}e$ (the Borel subalgebra) is not anti-invariant under ω.

2.2 Basic Definitions

Kac-Moody algebras are infinite-dimensional generalizations of the finite simple Lie algebras, as classified by Cartan and Killing. A standard treatment of an l-dimensional Lie algebra is in terms of a set of generators $\{T_m \mid m = 1, \ldots, l\}$, subject to the commutation relations

$$[T_m, T_n] = f_{mn}{}^p T_p, \tag{2.2.1}$$

where the structure constants $f_{mn}{}^p$ contain all the information of the algebra. This construction is not very convenient if we want to generalize it to cases when $l \to \infty$. It is for this reason that we in the previous section emphasized the importance of the Chevalley-Serre presentation, a construction which is amenable for generalization. We turn now to discuss the basic properties of Kac-Moody algebras, using the Chevalley-Serre presentation.

2.2.1 The Chevalley-Serre Presentation of $\mathfrak{g}(A)$

Let (e_i, f_i, h_i) be a triple of generators, satisfying the commutation relations of $\mathfrak{sl}(2,\mathbb{R})$:

$$[e_i, f_i] = h_i, \qquad [h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i. \tag{2.2.2}$$

We can now construct an algebra $\tilde{\mathfrak{g}}$ by letting the index i run from 1 to r and "intertwining" the r copies of $\mathfrak{sl}(2,\mathbb{R})$ through the following relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j, \\ [h_i, e_j] &= A_{ij} e_j, \\ [h_i, f_j] &= -A_{ij} f_j, \\ [h_i, h_j] &= 0. \end{aligned} \tag{2.2.3}$$

It is important to note that here r is always finite, even if the algebra itself might be infinite-dimensional. The structure of the algebra $\tilde{\mathfrak{g}}$ is encoded in the *Cartan matrix* A,

whose entries, A_{ij}, determine the commutation relations between the generators of the different $\mathfrak{sl}(2,\mathbb{R})$-subalgebras. We shall discuss the Cartan matrix in more detail below, and for now we just impose that it be non-degenerate, $\det A \neq 0$. This condition will be lifted later on. We shall often emphasize the dependence of $\tilde{\mathfrak{g}}$ on the Cartan matrix, and write $\tilde{\mathfrak{g}}(A)$. From now on we also fix the base field of $\tilde{\mathfrak{g}}$ to $\mathbb{R}$.

Further elements of $\tilde{\mathfrak{g}}$ are obtained by taking multiple commutators as follows

$$[e_{i_1},[e_{i_2},\ldots,[e_{i_{k-1}},e_{i_k}]\cdots]], \qquad [f_{i_1},[f_{i_2},\ldots,[f_{i_{k-1}},f_{i_k}]\cdots]]. \tag{2.2.4}$$

Note that at this point the algebra $\tilde{\mathfrak{g}}$ is infinite-dimensional because there are no relations between the e_i or f_i that restrict these multicommutators.

Through the use of the Chevalley relations, (2.2.3), any commutator involving the h_i may be reduced to one of the form of (2.2.4). This gives rise to the so-called *triangular decomposition* of $\tilde{\mathfrak{g}}$, which takes the form (direct sum of vector spaces)

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+. \tag{2.2.5}$$

Here, $\mathfrak{h}$ is a real vector space spanned by the h_i,

$$\mathfrak{h} = \sum_{i=1}^{r} \mathbb{R} h_i, \tag{2.2.6}$$

which forms an abelian subalgebra of $\tilde{\mathfrak{g}}$, called the *Cartan subalgebra*. The subspaces $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are freely generated by the e_i's and the f_i's, respectively. The algebra $\tilde{\mathfrak{g}}$ is not simple, but has a maximal ideal $\mathfrak{i}$, which decomposes as a direct sum of ideals,

$$\mathfrak{i} = (\mathfrak{i}\cap\tilde{\mathfrak{n}}_-) \oplus (\mathfrak{i}\cap\tilde{\mathfrak{n}}_+) \equiv \mathfrak{i}_- \oplus \mathfrak{i}_+, \tag{2.2.7}$$

where the two subspaces $\mathfrak{i}_+$ and $\mathfrak{i}_-$ are ideals in $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$, respectively. It follows that we have

$$\mathfrak{i}_\pm \cap \mathfrak{h} = 0, \qquad \mathfrak{i}_+ \cap \mathfrak{i}_- = 0. \tag{2.2.8}$$

The two ideals $\mathfrak{i}_\pm$ are generated by the subsets of elements $S_\pm$ given by

$$\begin{aligned} S_+ &= \left\{ \mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) \;\middle|\; i\neq j,\ i,j=1,\ldots,r \right\}, \\ S_- &= \left\{ \mathrm{ad}_{f_i}^{1-A_{ij}}(f_j) \;\middle|\; i\neq j,\ i,j=1,\ldots,r \right\}, \end{aligned} \tag{2.2.9}$$

where "ad" denotes the adjoint action, i.e., for $x,y\in\tilde{\mathfrak{g}}$, $\mathrm{ad}_x(y)=[x,y]$. We shall now take the quotient of $\tilde{\mathfrak{g}}(A)$ by the ideal $\mathfrak{i}$. This gives rise to relations among the generators of $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$. We thereby define the *Kac-Moody algebra* $\mathfrak{g}(A)$, associated with the Cartan matrix A, as follows

$$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A) \;/\; \mathfrak{i}. \tag{2.2.10}$$

Since we have chosen the base field to be $\mathbb{R}$, this defines the split real form of the corresponding Kac-Moody algebra over $\mathbb{C}$. In the rest of this paper we will exclusively be working with split real forms.

The rank r Kac-Moody algebra $\mathfrak{g}(A)$ is now generated by the $3r$ generators e_i, f_i, h_i subject to the Chevalley relations, (2.2.3), and the *Serre relations*,

$$\begin{aligned} \mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) &= [e_i,[e_i,\ldots,[e_i,e_j]\cdots]] = 0, \\ \mathrm{ad}_{f_i}^{1-A_{ij}}(f_j) &= [f_i,[f_i,\ldots,[f_i,f_j]\cdots]] = 0, \end{aligned} \tag{2.2.11}$$

with each relation containing $1-A_{ij}$ commutators. The triangular decomposition of $\mathfrak{g}(A)$ then reads

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{2.2.12}$$

where

$$\mathfrak{n}_\pm = \tilde{\mathfrak{n}}_\pm \;/\; \mathfrak{i}_\pm. \tag{2.2.13}$$

The Serre relations thus impose restrictions on $\mathfrak{n}_\pm$ which cut the chains of multiple commutators involving e_i and f_i. These restrictions might, or might not, render the algebra $\mathfrak{g}(A)$ finite-dimensional. We shall see in the next section how this depends on the properties of the Cartan matrix.

2.2.2 The Cartan Matrix and Dynkin Diagrams

So far we have discussed how the algebra $\mathfrak{g}(A)$ is constructed by imposing relations between the Chevalley generators e_i, f_i, h_i. These relations are completely determined by the entries of the matrix A, an important object which we shall now discuss more closely.

An $r \times r$ matrix $A = (A_{ij})_{i,j=1,\dots,r}$ is called a *generalized Cartan matrix* if it satisfies the following properties:

$$\begin{aligned} A_{ii} &= 2, \quad i = 1, \dots, r, \\ A_{ij} &= 0 \Leftrightarrow A_{ji} = 0, \\ A_{ij} &\in \mathbb{Z}_- \quad (i \neq j). \end{aligned} \tag{2.2.14}$$

For brevity, we shall in the following refer to A simply as a Cartan matrix. The Cartan matrix is called *indecomposable* if the index set $\mathcal{S} = \{1, \dots, r\}$ can not be divided into two non-empty subsets $\mathcal{I}$ and $\mathcal{J}$ such that $A_{ij} = 0$ for $i \in \mathcal{I}$ and $j \in \mathcal{J}$. An important statement is then the following: *when the Cartan matrix A is non-degenerate, det $A \neq 0$, and indecomposable, the Kac-Moody algebra $\mathfrak{g}(A)$ is simple* [15]. In the following we shall always assume that A is indecomposable.

It is now possible to provide a (partial) classification of the various types of algebras $\mathfrak{g}(A)$ that can be constructed from a Cartan matrix. There exist three main classes:

- If A is positive definite, the algebra $\mathfrak{g}(A)$ is finite-dimensional and falls under the Cartan-Killing classification, i.e., it is one of the finite simple Lie algebras $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7$ or E_8.
- If A is positive-semidefinite, i.e., det $A = 0$ with one zero eigenvalue, the algebra is infinite-dimensional and is said to be an *affine* Kac-Moody algebra.[1] All affine Kac-Moody algebras are classified [15].
- If A is not part of the two classes above, the algebra $\mathfrak{g}(A)$ is infinite-dimensional and is generally called an *indefinite* Kac-Moody algebra, by virtue of the fact that A is of indefinite signature.

For the third class above, no general classification exists. We shall, however, mainly be interested in a subclass of the indefinite Kac-Moody algebras, corresponding to the case when the matrix A has one negative eigenvalue and $r-1$ positive eigenvalues. The associated Kac-Moody algebras are called *Lorentzian*, because of the signature $(-++\cdots++)$ of A. A special subclass of the Lorentzian algebras, known as *hyperbolic* Kac-Moody algebras, have in fact been classified. We shall define hyperbolic Kac-Moody algebras in Section 2.4.2.

Since most of the entries of the Cartan matrix are zero, it is convenient to encode the non-vanishing entries in a diagram, $\Gamma = \Gamma(A)$, called a *Dynkin diagram.* To this end, we associate a node $\circ$ in Γ to each Chevalley triple (e_i, f_i, h_i), and if $A_{ij} \neq 0$ for $i \neq j$ the nodes i and j are connected by $\max(|A_{ij}|, |A_{ji}|)$ lines. In addition, when $|A_{ij}| > |A_{ji}|$ we draw an arrow from node j to node i. Indecomposability of the Cartan matrix A is equivalent to the statement that the Dynkin diagram $\Gamma(A)$ is connected.

[1] Strictly speaking, in the case of det $A = 0$ the algebra constructed from the Chevalley-Serre relations only corresponds to the *derived* algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. We shall come back to this issue in Section 2.3.2.

Let us now discuss some simple examples in order to illustrate the relation between the Kac-Moody algebra $\mathfrak{g}(A)$, its Cartan matrix A and the associated Dynkin diagram $\Gamma(A)$. We begin with the simplest possible case, namely the Lie algebra $A_1 = \mathfrak{sl}(2,\mathbb{R})$, discussed at length in Section 2.1. Here there is only one Chevalley triple, (e, f, h), and consequently the Cartan matrix is just the number (2), with Dynkin diagram consisting of one node $\circ$. The Lie algebra $A_2 = \mathfrak{sl}(3,\mathbb{R})$, in turn, is described by the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and Dynkin diagram $\circ\!\!-\!\!\!-\!\!\circ$. This corresponds to two copies of $\mathfrak{sl}(2,\mathbb{R})$ which are intertwined through the non-vanishing off-diagonal components of the Cartan matrix. In contrast, if $A_{12} = A_{21} = 0$ we have a direct sum of Lie algebras $A_1 \oplus A_1$ corresponding to the decomposable Cartan matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

with Dynkin diagram $\circ\ \ \circ$. This algebra is not simple, since the two A_1's constitute two non-trivial ideals. Later on we shall discuss these examples, and more involved ones, in more detail.

2.2.3 The Root System and the Root Lattice

A very important notion in the theory of Kac-Moody algebras is that of a *root*. In this section we shall develop the basic theory of roots and examine the vector space which they span, hopefully convincing the reader that these issues are extremely useful for a deeper understanding of Kac-Moody algebras.

Let us begin by noting that, by virtue of the Chevalley relations, (2.2.3), the adjoint action of $\mathfrak{h}$ on $\mathfrak{n}_\pm$ is diagonal,

$$\mathrm{ad}_h(e_i) = [h, e_i] = \alpha_i(h) e_i, \qquad h \in \mathfrak{h}, \tag{2.2.15}$$

and similarly for the action on f_i. The eigenvalue $\alpha_i(h)$ represents the value of a linear map from $\mathfrak{h}$ to the real numbers,

$$\alpha_i \ :\ \mathfrak{h} \ \ni\ h \ \longmapsto\ \alpha_i(h) \ \in\ \mathbb{R}. \tag{2.2.16}$$

The linear maps α_i are called *simple roots* and belong to the dual space $\mathfrak{h}^\star$. We shall sometimes employ the notation $\langle \alpha, h\rangle = \alpha(h)$ for the pairing between a form $\alpha \in \mathfrak{h}^\star$ and a vector $h \in \mathfrak{h}$. If the eigenvalue $\alpha(h)$ vanishes, then α is not a root. It is also common to refer to the Cartan generators h_i as *simple coroots* to emphasize that they belong to the dual of the space of roots. In this case one also writes $\alpha_i^\vee \equiv h_i$. The same analysis can be performed for multiple commutators, e.g,

$$\begin{aligned} [h, [e_i, e_j]] &= -[e_i, [e_j, h]] - [e_j, [h, e_i]] \\ &= (\alpha_i + \alpha_j)(h)[e_i, e_j], \end{aligned} \tag{2.2.17}$$

where in the first line we made use of the Jacobi identity. If the generator $e_\alpha \equiv [e_i, e_j]$ is non-vanishing, i.e., is not killed by the Serre relations, then $\alpha \equiv \alpha_i + \alpha_j$ is the root associated with e_α. We denote by $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ the basis of simple roots and by $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ the basis of simple coroots. Any root can be expressed as an integer linear combination of the simple roots. We denote by Φ the complete set of roots. This is called the *root system*. In analogy with (2.2.6) we also have

$$\mathfrak{h}^\star = \sum_{i=1}^{r} \mathbb{R}\alpha_i. \tag{2.2.18}$$

A root is called *positive* (*negative*) if it can be written as a linear combination of the simple roots Π with only non-negative (non-positive) coefficients. From the triangular decomposition it follows that all roots are either positive or negative. Thus, the root system, Φ, splits into a disjoint union of positive and negative roots,

$$\Phi = \Phi_+ \cup \Phi_-. \tag{2.2.19}$$

For $\mathfrak{g} \ni x_\alpha \neq 0$ the associated root

$$\alpha = \sum_{i=1}^{r} m_i \alpha_i \tag{2.2.20}$$

belongs to Φ_+ if all $m_i \in \mathbb{Z}_{\geq 0}$ and to Φ_- if all $m_i \in \mathbb{Z}_{\leq 0}$. Let us for definiteness take α to be a positive root. Then $-\alpha$ is necessarily a negative root, and we write

$$x_\alpha \equiv e_\alpha \in \mathfrak{n}_+, \tag{2.2.21}$$

$$x_{-\alpha} \equiv f_\alpha \in \mathfrak{n}_-. \tag{2.2.22}$$

In the Chevalley basis the eigenvalues $\alpha_i(h)$ are always integers, called the *Cartan integers*, revealing that the set of roots Φ lie on an r-dimensional lattice Q spanned by the simple roots,

$$Q = \sum_{i=1}^{r} \mathbb{Z}\alpha_i \subset \mathfrak{h}^\star. \tag{2.2.23}$$

All elements of the root system thus belong to Q but the converse is not true,

$$\Phi \subset Q. \tag{2.2.24}$$

In the Cartan subalgebra $\mathfrak{h}$ we similarly have the dual notion of a *coroot lattice*:

$$Q^\vee = \sum_{i=1}^{r} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}. \tag{2.2.25}$$

We may now decompose the algebra $\mathfrak{g}$ into disjoint subsets $\mathfrak{g}_\alpha \subset \mathfrak{g}$, where each subset is spanned by those generators $x \in \mathfrak{g}$, whose eigenvalue under the action of $h \in \mathfrak{h}$ is given by $\alpha(h)$. This decomposition is called the *root space decomposition* and reads

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{2.2.26}$$

where the subspace $\mathfrak{g}_\alpha$ is the *root space* associated to the root α. Explicitly these are given by

$$\mathfrak{g}_\alpha = \big\{ x \in \mathfrak{g} \,\big|\, \forall\, h \in \mathfrak{h} \;:\; \mathrm{ad}_h(x) = \alpha(h)x \big\}. \tag{2.2.27}$$

Note that the zeroth subspace $\mathfrak{g}_0$ coincides with the Cartan subalgebra, $\mathfrak{g}_0 = \mathfrak{h}$. Because of the disjoint split $\Phi_+ \cup \Phi_-$ of the root system we can write the root space decomposition as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_{-\alpha}. \tag{2.2.28}$$

The dimension of each subspace $\mathfrak{g}_\alpha$ is called the *multiplicity*, $\mathrm{mult}(\alpha)$, of the root α,

$$\mathrm{mult}(\alpha) \equiv \dim\, \mathfrak{g}_\alpha. \tag{2.2.29}$$

Thus, for a given root $\gamma \in \Phi$, with root space

$$\mathfrak{g}_\gamma = \mathbb{R}x_\gamma^{(1)} \oplus \mathbb{R}x_\gamma^{(2)} \oplus \cdots \oplus \mathbb{R}x_\gamma^{(k-1)} \oplus \mathbb{R}x_\gamma^{(k)}, \tag{2.2.30}$$

we have

$$\mathrm{mult}(\gamma) = k \;\in\; \mathbb{Z}_+\backslash\{0\}. \tag{2.2.31}$$

The root spaces corresponding to the simple roots are one-dimensional

$$\mathfrak{g}_{\alpha_i} = \mathbb{R}e_i, \qquad \mathfrak{g}_{-\alpha_i} = \mathbb{R}f_i, \tag{2.2.32}$$

and, consequently, the multiplicities of the simple roots are one,

$$\mathrm{mult}(\alpha_i) = 1. \tag{2.2.33}$$

For finite-dimensional Lie algebras the root multiplicities are always one. This does not carry over to infinite-dimensional Kac-Moody algebras, for which roots can have arbitrarily large multiplicity. We shall come back to the issue of root multiplicities in Section 2.2.6 when we discuss the Weyl group. We can now write the full root system as follows

$$\Phi = \{\alpha \in \mathfrak{h}^\star \mid \alpha \neq 0,\ \mathfrak{g}_\alpha \neq 0\}. \tag{2.2.34}$$

A useful notion is that of the *height* of a root. This is a linear integral map

$$\mathrm{ht} : \alpha \;\longmapsto\; \mathrm{ht}(\alpha) \in \mathbb{Z} \tag{2.2.35}$$

defined as the sum of the coefficients of α in the basis of simple roots (see (2.2.20)),

$$\mathrm{ht}(\alpha) = \sum_{i=1}^{r} m_i. \tag{2.2.36}$$

It follows that for $\alpha \in \Phi_+$ we have $\mathrm{ht}(\alpha) > 0$, and vice versa for the negative roots.

We now have a better understanding of the appearance of the Cartan matrix in (2.2.3). It simply corresponds to the values of the simple roots $\alpha_j \in \mathfrak{h}^\star$ acting on the simple coroots $\alpha_i^\vee \in \mathfrak{h}$, i.e.,

$$A_{ij} = \alpha_j(\alpha_i^\vee) = \langle \alpha_j, \alpha_i^\vee \rangle\,. \tag{2.2.37}$$

Finally we shall here develop a more geometric description of the root system, which is very useful for our understanding of the Kac-Moody algebra $\mathfrak{g}(A)$. An arbitrary root $\gamma \in \Phi$ may be seen as a "vector" in $\mathfrak{h}^\star$ with components given by

$$\gamma_i \equiv \gamma(h_i), \tag{2.2.38}$$

i.e., the components of γ correspond to the different values of the root γ acting on the simple coroots $h_i = \alpha_i^\vee$. We shall sometimes write the *root vector* $\vec{\gamma}$ as

$$\vec{\gamma} = (\gamma_1, \ldots, \gamma_r) \;\in\; \mathfrak{h}^\star. \tag{2.2.39}$$

From this point of view the entries A_{ij} of the Cartan matrix correspond to the components of the root vectors $\vec{\alpha}_i$ associated with the simple roots:

$$\begin{aligned} \vec{\alpha}_i &= (\alpha_{i(1)}, \alpha_{i(2)}, \ldots, \alpha_{i(r)}) \\ &= (A_{1i}, A_{2i}, \ldots, A_{ri}), \end{aligned} \tag{2.2.40}$$

where we have indicated the component index (i) of the simple roots within parenthesis to distinguish it from the index i labeling the different simple roots.

We conclude this section by defining an involution ω on the Kac-Moody algebra $\mathfrak{g}(A)$, known as the *Chevalley involution*. This is defined as follows on the Chevalley generators:

$$\omega(e_i) = -f_i, \qquad \omega(f_i) = e_i, \qquad \omega(h_i) = -h_i. \tag{2.2.41}$$

This involution leaves the Chevalley relations, (2.2.3), invariant and therefore corresponds to an automorphism of $\mathfrak{g}(A)$. The involution is multilinear when acting on multiple commutators, e.g, on $e_3 \equiv [e_1, e_2] \in \mathfrak{n}_+ \subset \mathfrak{g}(A)$ it acts as

$$\omega(e_3) = \omega([e_1, e_2]) = [\omega(e_1), \omega(e_2)] = [f_1, f_2] = f_3 \in \mathfrak{n}_- \subset \mathfrak{g}(A). \tag{2.2.42}$$

The subset of $\mathfrak{g}(A)$ which is pointwise fixed under ω defines the *maximal compact subalgebra*

$$K(\mathfrak{g}) = \big\{x \in \mathfrak{g}(A) \bigm| \omega(x) = x\big\} \ \subset \ \mathfrak{g}. \tag{2.2.43}$$

The maximal compact subalgebra is generated by the combinations $e_i - f_i,\ i = 1, \ldots, r$, of Chevalley generators. We have the induced *Cartan decomposition* of $\mathfrak{g}(A)$ (direct sum of vector spaces):

$$\mathfrak{g} = K(\mathfrak{g}) \oplus \mathfrak{p}, \tag{2.2.44}$$

where the complement $\mathfrak{p}$ is the subset of $\mathfrak{g}$ which is pointwise anti-invariant under ω,

$$\mathfrak{p} = \big\{x \in \mathfrak{g}(A) \bigm| \omega(x) = -x\big\}. \tag{2.2.45}$$

This is not a subalgebra of $\mathfrak{g}$, but elements of $\mathfrak{p}$ transforms in some representation of $K(\mathfrak{g})$. The Cartan decomposition yields the following characteristic properties of a *symmetric space*:

$$[\mathfrak{p}, \mathfrak{p}] \subset K(\mathfrak{g}), \qquad [K(\mathfrak{g}), \mathfrak{p}] \subset \mathfrak{p}, \qquad [K(\mathfrak{g}), K(\mathfrak{g})] \subset K(\mathfrak{g}). \tag{2.2.46}$$

Let us also note here an additional important decomposition of $\mathfrak{g}(A)$. This is the *Iwasawa decomposition* which reads

$$\mathfrak{g} = K(\mathfrak{g}) \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{2.2.47}$$

In the finite-dimensional case this decomposition reduces to the familiar fact that any matrix can be decomposed into an orthogonal part, a diagonal part and an upper triangular part. The subset

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+ \tag{2.2.48}$$

is known as the *Borel subalgebra*. The Iwasawa decomposition will be of importance in Sections 3.2.2 and 3.6.

2.2.4 The Invariant Bilinear Form

To proceed with the analysis of the roots of a Kac-Moody algebra it is useful to first define a "metric" $(\cdot|\cdot)$ on the space $\mathfrak{h}^\star$. This will then be extended to an invariant bilinear form on the entire Kac-Moody algebra $\mathfrak{g}(A)$, and will thereby also play an important role in many of the subsequent developments.

We shall assume, as before, that the Cartan matrix, is non-degenerate, and, in addition, we shall take it to be *symmetrizable*. The first condition will be lifted later on, while the second condition will be kept throughout the remainder of these lectures. Symmetrizability of A implies that there exists a diagonal matrix $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_r)$, with all $\epsilon_i > 0$, such that the Cartan matrix decomposes according to

$$A = DS, \tag{2.2.49}$$

where S is a symmetric $r \times r$ matrix. The matrix $S = (S_{ij})$ now defines a symmetric invertible bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^\star$ as follows

$$S_{ij} \equiv (\alpha_i|\alpha_j), \tag{2.2.50}$$

for $\alpha_i, \alpha_j \in \Pi$. Moreover, by imposing the defining relation $A_{ii} = 2$ we find

$$\epsilon_i = \frac{2}{(\alpha_i|\alpha_i)}. \tag{2.2.51}$$

We have now defined a bilinear form on the space $\mathfrak{h}^\star$, which in turn induces a bilinear form on the root lattice Q. An important consequence of this is the following:

- For finite-dimensional Lie algebras the bilinear form $(\cdot|\cdot)$ is of Euclidean signature and, consequently, Q is a Euclidean lattice. In this case the bilinear form coincides with the standard Killing form.

- For Lorentzian Kac-Moody algebras the bilinear form $(\cdot|\cdot)$ is a flat metric with signature $(-+\cdots+)$ and, consequently, Q is a Lorentzian lattice.

The bilinear form can now be extended to the full Kac-Moody algebra. Since $(\cdot|\cdot)$ is non-degenerate it defines an isomorphism $\mu : \mathfrak{h}^\star \to \mathfrak{h}$ as follows:

$$\langle \alpha, \mu(\beta)\rangle \equiv (\alpha|\beta), \qquad \beta, \alpha \in \mathfrak{h}^\star,\ \mu(\beta) \in \mathfrak{h}, \tag{2.2.52}$$

with the inverse map $\mu^{-1} : \mathfrak{h} \to \mathfrak{h}^\star$ then defines a bilinear form $(\cdot|\cdot)$ on the Cartan subalgebra $\mathfrak{h}$ through

$$\left\langle \mu^{-1}(\alpha^\vee), \beta^\vee \right\rangle \equiv (\alpha^\vee|\beta^\vee), \qquad \alpha^\vee, \beta^\vee \in \mathfrak{h},\ \mu(\alpha^\vee) \in \mathfrak{h}^\star. \tag{2.2.53}$$

Then, from the definition of $(\cdot|\cdot)$ in (2.2.50), we find

$$(\alpha_i|\beta) = \frac{1}{\epsilon_i}\left\langle \beta, \alpha_i^\vee \right\rangle. \tag{2.2.54}$$

In addition, by virtue of (2.2.52), we have the relation

$$(\alpha_i|\beta) = \langle \beta, \mu(\alpha_i)\rangle, \tag{2.2.55}$$

and by equating (2.2.54) with (2.2.55) we arrive at the explicit expressions

$$\mu(\alpha_i) = \frac{1}{\epsilon_i}\alpha_i^\vee \qquad \text{or} \qquad \mu^{-1}(\alpha_i^\vee) = \epsilon_i \alpha_i. \tag{2.2.56}$$

We can use this result to find a relation between the bilinear forms on $\mathfrak{h}$ and $\mathfrak{h}^\star$:

$$(\alpha_i^\vee|\alpha_j^\vee) = \epsilon_i\epsilon_j(\alpha_i|\alpha_j), \tag{2.2.57}$$

and in the special case $i = j$ we thus have

$$\frac{(\alpha_i^\vee|\alpha_i^\vee)}{2} = \frac{2}{(\alpha_i|\alpha_i)}. \tag{2.2.58}$$

Let us further note that (2.2.54) ensures that the Cartan matrix can be expressed solely in terms of the bilinear form

$$A_{ij} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}, \tag{2.2.59}$$

an expression which is very useful for practical purposes.

At this point we have a non-degenerate symmetric bilinear form on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}(A)$. To extend this to the entire algebra, one exploits the *invariance* of $(\cdot|\cdot)$, i.e., the property

$$([x,y],z) = (x,[y,z]), \qquad x,y,z \in \mathfrak{g}(A). \tag{2.2.60}$$

For example, by computing $(\alpha_i^\vee|[\alpha_k^\vee, e_j]) = A_{kj}(\alpha_i^\vee|e_j)$ and using the invariance on the left hand side we find

$$(\alpha_i^\vee|e_j) = 0, \tag{2.2.61}$$

because of the fact that $[\alpha_i^\vee, \alpha_k^\vee] = 0$. A similar argument gives $(\alpha_i^\vee|f_j) = 0$. Moreover, by computing $([\alpha_k^\vee, e_i]|f_j)$ one finds

$$(e_i|f_j) = \epsilon_i\delta_{ij}, \tag{2.2.62}$$

or, more generally, for two arbitrary generators $x_\alpha, x_\beta \in \mathfrak{g}(A)$ one has

$$(x_\alpha|x_\beta) \sim \delta_{\alpha,-\beta}, \tag{2.2.63}$$

where the proportionality constant depends on the normalization of the Chevalley generators.

Before we proceed with some examples, let us discuss some additional features of the root system Φ of a Kac-Moody algebra $\mathfrak{g}(A)$. In the special case when $\mathfrak{g}(A)$ is a rank r finite-dimensional Lie algebra we have seen that the root lattice is an r-dimensional Euclidean lattice, thus implying that all roots have positive norm, $\alpha^2 > 0$, $\forall \alpha \in \Phi$. In the general case however, the root lattice can have arbitrary signature, and thus roots can in general have positive, zero or negative norm. We shall adopt the standard terminology and call roots of positive norm *real* roots, and those of zero or negative norm, *imaginary* roots. In this way the root system of a Kac-Moody algebra decomposes into two disjoint sets $\Phi_{\Re}$ and $\Phi_{\Im}$ of real and imaginary roots, respectively. The largest norm squared of the real roots is, by analogy with the finite-dimensional case, restricted to 2, while the imaginary roots can come with arbitrarily large negative norm squared. We can thus describe these two types of roots as follows

$$\begin{aligned} \Phi_{\Re} &= \big\{\alpha \in \Phi \;\big|\; 0 < (\alpha|\alpha) \leq 2\big\}, \\ \Phi_{\Im} &= \big\{\beta \in \Phi \;\big|\; (\beta|\beta) \leq 0\big\}, \end{aligned} \tag{2.2.64}$$

and we have

$$\Phi = \Phi_{\Im} \cup \Phi_{\Re}. \tag{2.2.65}$$

The multiplicity of the real roots is always one, $\text{mult}(\alpha) = 1$, $\forall \alpha \in \Phi_{\Re}$, while the imaginary roots generally come with a non-trivial multiplicity, $\text{mult}(\beta) > 1$, $\forall \beta \in \Phi_{\Im}$. In particular, for the indefinite Kac-Moody algebras the multiplicity of the imaginary roots grows exponentially with increasing height, thus rendering these algebras very difficult to control. We shall come back to the issue of real and imaginary roots in Section 2.2.6 after we have learned some of the basic properties of the Weyl group.

2.2.5 Example: A_2 versus A_1^+

Let us now try to make all this a bit more concrete, by introducing and comparing two examples in detail. We shall consider the familiar finite-dimensional Lie algebra $A_2 = \mathfrak{sl}(3,\mathbb{R})$ and the infinite-dimensional affine Kac-Moody algebra A_1^+ (the notation will be explained in Section 2.4.1). Our goal is firstly to understand what it is that makes the first one finite and the second one infinite-dimensional. Secondly, we shall investigate and compare the two different root systems.

THE SERRE RELATIONS

The rank 2 Lie algebras A_2 and A_1^+ are described by the Cartan matrices

$$A[A_2] = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad A[A_1^+] = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \tag{2.2.66}$$

with the associated Dynkin diagrams displayed in Figure 2.1. For simplicity of notation, we shall refer to the two different Cartan matrices simply as $A = A[A_2]$ and $\bar{A} = A[A_1^+]$.

Figure 2.1: On the left the Dynkin diagram of the Lie algebra $A_2 = \mathfrak{sl}(3,\mathbb{R})$ and on the right the Dynkin diagram of the affine Kac-Moody algebra A_1^+.

The Chevalley generators for A_2 are $\{e_1, e_2, f_1, f_2, h_1, h_2\}$ and the ones for A_1^+ are $\{\bar{e}_1, \bar{e}_2, \bar{f}_1, \bar{f}_2, \bar{h}_1, \bar{h}_2\}$. The commutation relations follow from the general form of the

Chevalley relations in (2.2.3), with the insertion of the individual Cartan matrix components. For example, we have

$$A_2 \quad : \quad [h_1, h_2] = 0, \quad [h_1, e_2] = -e_2, \quad [h_2, e_1] = -e_1, \quad [e_1, f_1] = h_1,$$

$$A_1^+ \quad : \quad [\bar{h}_1, \bar{h}_2] = 0, \quad [\bar{h}_1, \bar{e}_2] = -2\bar{e}_2, \quad [\bar{h}_2, \bar{e}_1] = -2\bar{e}_1, \quad [\bar{e}_1, \bar{f}_1] = \bar{h}_1. \tag{2.2.67}$$

It is clear that the relations in A_2 are remarkably similar to those in A_1^+ with the only difference arising in the relations involving the off-diagonal entries of the Cartan matrices, which, of course, is the only obvious distinction between the algebras at this point. We now want to understand how this seemingly trivial change in the Cartan matrix can render an algebra infinite-dimensional. The answer lies in the Serre relations.

Let us now proceed to check the Serre relations involving the positive nilpotent generators of "e-type". The analysis is analogous for the negative ones. For A_2 we then have

$$\mathrm{ad}_{e_1}^{1-A_{12}}(e_2) = [e_1, [e_1, e_2]] = 0, \tag{2.2.68}$$

implying that the generator $[e_1, [e_1, e_2]]$ does not exist in the algebra. The commutators $e_3 \equiv [e_1, e_2]$, on the other hand, is not killed by (2.2.68) and so corresponds to a new generator. On the negative side, we similarly find the new generator $f_3 \equiv -[f_1, f_2]$. No other non-vanishing generators exist in A_2 and therefore the algebra is eight-dimensional. We may take as a basis of A_2 the eight elements $\{e_1, e_2, e_3, f_1, f_2, f_3, h_1, h_2\}$. This corresponds to the adjoint representation $\mathbf{8}$ of $\mathfrak{sl}(3, \mathbb{R})$.

We now turn to A_1^+. The Serre relation for $\bar{e}_1$ and $\bar{e}_2$ reads

$$\mathrm{ad}_{\bar{e}_1}^{1-\bar{A}_{12}}(\bar{e}_2) = [\bar{e}_1, [\bar{e}_1, [\bar{e}_1, \bar{e}_2]]] = 0. \tag{2.2.69}$$

This condition therefore kills the generator $[\bar{e}_1, [\bar{e}_1, [\bar{e}_1, \bar{e}_2]]]$ in A_1^+, while there are no restrictions on the following two generators:

$$\bar{e}_3 \equiv [\bar{e}_1, \bar{e}_2], \qquad \bar{e}_4 = [\bar{e}_1, [\bar{e}_1, \bar{e}_2]]. \tag{2.2.70}$$

In addition, we have the Serre relation for $\bar{e}_2$ acting on $\bar{e}_1$ which yields yet another non-vanishing generator

$$\bar{e}_5 \equiv [\bar{e}_2, [\bar{e}_2, \bar{e}_1]]. \tag{2.2.71}$$

It is the existence of $\bar{e}_4$ and $\bar{e}_5$ which renders A_1^+ infinite-dimensional. For example, consider the following multicommutator, alternating between $\bar{e}_1$ and $\bar{e}_2$,

$$[\bar{e}_1, [\bar{e}_2, [\bar{e}_1, \bar{e}_2]]] = -[\bar{e}_1, [\bar{e}_2, [\bar{e}_2, \bar{e}_1]] \neq 0. \tag{2.2.72}$$

In A_2 this commutator would have been zero because the Serre relations impose

$$[e_2, [e_2, e_1]] = 0,$$

while in A_1^+ it corresponds to $-[\bar{e}_1, \bar{e}_5]$ which is unrestricted. It is possible to continue in this way, and any alternating multicommutator is non-vanishing, e.g,

$$[\bar{e}_1, [\bar{e}_2, [\bar{e}_1, [\bar{e}_2, \ldots, [\bar{e}_1, \bar{e}_2] \cdots]]]] \in A_1^+. \tag{2.2.73}$$

THE ROOT SYSTEMS

We shall now proceed to compare the two algebras A_2 and A_1^+ at the level of their respective systems of roots. To A_2 we associate the simple roots $\Pi = \{\alpha_1, \alpha_2\}$, and to A_1^+, $\bar{\Pi} = \{\bar{\alpha}_1, \bar{\alpha}_2\}$. Since the Cartan matrices are symmetric, they give directly the bilinear forms on the space of roots. We have

$$\begin{aligned} &(\alpha_1|\alpha_1) = 2, \quad (\alpha_2|\alpha_2) = 2, \quad (\alpha_1|\alpha_2) = -1, \\ &(\bar{\alpha}_1|\bar{\alpha}_1) = 2, \quad (\bar{\alpha}_2|\bar{\alpha}_2) = 2, \quad (\bar{\alpha}_1|\bar{\alpha}_2) = -2. \end{aligned} \tag{2.2.74}$$

All roots can be described as integral non-negative or non-positive linear combinations of the simple roots. For A_2, we find that $\alpha_1 + \alpha_2$ is a root, because the corresponding generator $e_3 \equiv [e_1, e_2]$ survives the Serre relations. Thus we define $\alpha_3 \equiv \alpha_1 + \alpha_2$. Let is now try to add yet another simple root and take, say, $\alpha_3 + \alpha_2$. This corresponds to the generator $[e_3, e_2] = [[e_1, e_2], e_2]$ which is zero in A_2 because of the Serre relations. In this way we find that the root system $\Phi = \Phi(A)$ of A_2 is given by

$$\Phi = \Phi_+ \cup \Phi_- = \{\alpha_1, \alpha_2, \alpha_3\} \cup \{-\alpha_1, -\alpha_2, -\alpha_3\}, \tag{2.2.75}$$

revealing that indeed the root system of A_2 is finite. Of course, any vector $m\alpha_1 + n\alpha_2$, $m, n \in \mathbb{Z}$, lies on the root lattice Q of A_2, even though it is not a root. Because the root α_3 is the root with largest height, $\mathrm{ht}(\alpha_3) = 2$, of Φ, it is the highest root of the algebra. This is also the highest weight of the adjoint representation.

It is illuminating to draw the root system in a root diagram, which makes it easier to visualize the structure of the algebra. To this end we define, as described in the previous section, the simple root vectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$, with components

$$\begin{aligned} \vec{\alpha}_1 &= (A_{11}, A_{12}) = (2, -1), \\ \vec{\alpha}_2 &= (A_{21}, A_{22}) = (-1, 2). \end{aligned} \tag{2.2.76}$$

The two vectors $\vec{\alpha}_1$ and $\vec{\alpha}_2$ span a two-dimensional Euclidean lattice, with separating angle of $2\pi/3$. We have indicated the root diagram of A_2 in Figure 2.2. Let us now analyze

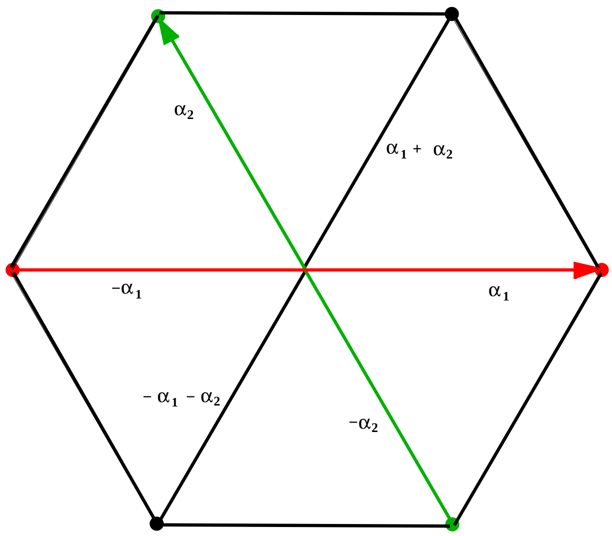

Figure 2.2: The root diagram of A_2, representing the adjoint representation **8**. The root $\alpha_1 + \alpha_2$ is the highest root corresponding to the highest weight of the representation.

the root system $\bar{\Phi} = \Phi(\bar{A})$ of A_1^+. We begin by noting that the determinant of the Cartan matrix vanishes

$$\det \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = 0, \tag{2.2.77}$$

as we have seen is the distinguishing feature of affine Kac-Moody algebras. This implies that the bilinear form of the algebra constructed from A is degenerate. We shall discuss how to deal with this feature in Section 2.3.2. For our present purposes, however, we just note that (2.2.77) implies that there exists a root $\bar{\delta} \in \bar{\Phi}$, which has zero norm,

$$(\bar{\delta}|\bar{\delta}) = 0. \tag{2.2.78}$$

In terms of the simple roots, we have

$$\bar{\delta} = \bar{\alpha}_1 + \bar{\alpha}_2, \tag{2.2.79}$$

as follows from (2.2.74). That $\bar{\delta}$ is indeed a root of $\bar{\Phi}$ can be seen by noting that the associated generator $[\bar{e}_1, \bar{e}_2]$ is non-vanishing. The existence of a null root is indicative of the fact that the algebra, as well as the associated root system, is infinite-dimensional.

In order to understand the root system it will prove convenient to write $\bar{\alpha}_2 = \bar{\delta} - \bar{\alpha}_1$, and treat the root system as a two-dimensional *Lorentzian* space, with basis vectors $\bar{\alpha}_1$ and $\bar{\delta}$. Is $\bar{\delta} - 2\bar{\alpha}_1$ a root of the algebra? The answer is no, because the associated generator vanishes,

$$\bar{e}_{\bar{\delta}-2\bar{\alpha}_1} \equiv [\bar{e}_{\bar{\delta}-\bar{\alpha}_1}, \bar{f}_1] = [\bar{e}_2, \bar{f}_1] = 0, \tag{2.2.80}$$

as follows from the Chevalley relations. However, $2\bar{\delta} - \bar{\alpha}_1 = \bar{\alpha}_1 + 2\bar{\alpha}_2$ is a root since it corresponds to the generator $[\bar{e}_2[\bar{e}_2, \bar{e}_1]] \neq 0$. One can iterate this procedure and find a complete description of the root system. All null roots are multiples of $\bar{\delta}$, while the real roots are combinations of $\pm\bar{\alpha}_1$ with $\bar{\delta}$. As discussed in the previous section, the root system thereby splits into disjoint sets corresponding to the *real* (i.e., spacelike) and the *imaginary* (i.e., lightlike) roots. Explicitly, the root system of A_1^+ then reads

$$\bar{\Phi} = \bar{\Phi}_{\Re} \cup \bar{\Phi}_{\Im} = \big\{ \pm \bar{\alpha}_1 + n\bar{\delta} \;\big|\; n \in \mathbb{Z} \big\} \cup \big\{ k\bar{\delta} \;\big|\; k \in \mathbb{Z} \setminus \{0\} \big\}. \tag{2.2.81}$$

In addition we have, of course, the usual split of $\bar{\Phi}$ into positive and negative roots.

2.2.6 The Weyl Group

A very important concept, which we shall use extensively in subsequent sections, is that of the *Weyl group* $\mathcal{W} = \mathcal{W}(A)$ of the Kac-Moody algebra $\mathfrak{g}(A)$. We begin by constructing the group $\mathcal{W}(A)$ abstractly and then we show how it is related to a Kac-Moody algebra. Fix a set of generators $\mathcal{S} = \{s_1, \dots, s_r\}$ and let $\tilde{\mathcal{W}}$ be the free group generated by $\mathcal{S}$. Let $m = (m_{ij})_{i,j=1,\dots,r}$ be an $r \times r$ matrix satisfying: (*i*) $m_{ii} = 1$, (*ii*) $m_{ij} \in \mathbb{Z}_{\geq 1},\ i \neq j$, and (*iii*) $m_{ij} = m_{ji}$. The group $\tilde{\mathcal{W}}$ then has a normal subgroup $\mathcal{N}$ generated by the particular combinations [16]

$$(s_i s_j)^{m_{ij}}. \tag{2.2.82}$$

The Weyl group $\mathcal{W}$, associated to the set $\mathcal{S}$, is the quotient group

$$\mathcal{W} \equiv \tilde{\mathcal{W}} \,/\, \mathcal{N}. \tag{2.2.83}$$

This is a particular instance of a *Coxeter group*, and the entries of the matrix (m_{ij}) are called *Coxeter exponents*. Our construction implies that the Weyl group, $\mathcal{W}$, is the group generated by the set $\mathcal{S}$ modulo the relations

$$(s_i s_j)^{m_{ij}} = 1, \quad i, j = 1, \dots, r. \tag{2.2.84}$$

The elements $s_i \in \mathcal{S}$ are called the *fundamental reflections*, by virtue of the property

$$s_i^2 = 1, \tag{2.2.85}$$

as follows from the first condition on (m_{ij}) above.

We now show how the group $\mathcal{W}(A)$ enters the story of Kac-Moody algebras. The Weyl group is a group of automorphisms of the root lattice Q,

$$\mathcal{W} \,:\, Q \longrightarrow Q, \tag{2.2.86}$$

with the fundamental reflections s_i being geometrically realized as reflections in the hyperplanes orthogonal to the simple roots α_i. More specifically, we associate a fundamental reflection s_i to each simple root α_i, such that the action on $\gamma \in Q$ is given by

$$s_i \,:\, \gamma \;\longmapsto\; s_i(\gamma) = \gamma - \langle \gamma, \alpha_i^\vee \rangle\, \alpha_i. \tag{2.2.87}$$

$A_{ij}A_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

Table 2.1: The relation between the entries A_{ij} of the Cartan matrix and the Coxeter exponents m_{ij}.

It is clear from this definition that s_i reverses the sign of α_i and pointwise fixes the hyperplane

$$T_i = \big\{\beta \in Q \;\big|\; \langle \beta, \alpha_i^\vee \rangle = 0\big\}. \tag{2.2.88}$$

Let us also check the condition $s_i^2 = 1$ explicitly for the geometric realization. Applying (2.2.87) twice on $\gamma \in Q$ yields

$$\begin{aligned} s_i \cdot s_i(\gamma) &= s_i\big(\gamma - \langle \gamma, \alpha_i^\vee \rangle\, \alpha_i\big) \\ &= \gamma - 2\,\langle \gamma, \alpha_i^\vee \rangle\, \alpha_i + \langle \gamma, \alpha_i^\vee \rangle\, \langle \alpha_i, \alpha_i^\vee \rangle\, \alpha_i \\ &= \gamma, \end{aligned} \tag{2.2.89}$$

where, in the last step, we made use of the fact that $\langle \alpha_i, \alpha_i^\vee \rangle = A_{ii} = 2$.

When realized geometrically in this way, the fundamental reflections are commonly called *Weyl reflections*. When acting on the simple roots themselves, the Weyl reflections become

$$s_i(\alpha_j) = \alpha_j - \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}\alpha_i = \alpha_j - A_{ij}\alpha_i, \tag{2.2.90}$$

where (A_{ij}) is the Cartan matrix. There is a simple relation between the entries of the Cartan matrix and the associated Coxeter exponents. This is displayed in Table 2.1. When $A_{ij}A_{ji} \geq 4$ the corresponding Coxeter exponents are infinite, implying that there are no relations between the generators s_i and s_j for these particular values of i and j.

The bilinear form $(\cdot|\cdot)$ is $\mathcal{W}$-invariant,

$$\big(\omega(\beta)\big|\omega(\beta')\big) = \big(\beta\big|\beta'\big), \tag{2.2.91}$$

which follows by direct calculation using (2.2.87). This implies that the Weyl group is "orthogonal" with respect to the bilinear form $(\cdot|\cdot)$ and hence is a discrete subgroup of the isometry group $O(\mathfrak{h}^\star)$ of $\mathfrak{h}^\star$,

$$\mathcal{W} \subset O(\mathfrak{h}^\star). \tag{2.2.92}$$

We can now make use of the Weyl group to get a better handle on the structure of the root system. The first important fact is that the root system $\Phi(A)$ of a Kac-Moody algebra $\mathfrak{g}(A)$ is $\mathcal{W}(A)$-invariant,

$$\mathcal{W} \cdot \Phi = \Phi. \tag{2.2.93}$$

We can associate a general Weyl reflection ω_α to any root $\alpha \in \Phi$. This will be described by a finite product of the fundamental reflections,

$$\omega_\alpha = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{i_k}, \tag{2.2.94}$$

where the minimal number k of fundamental reflections needed to describe ω_α is called the *length* of ω_α, and is denoted by $\ell(\alpha)$. By definition, the fundamental reflections have length one, $\ell(s_i) = 1$.

The reflection $\omega_\alpha \in \mathcal{W}$ fixes the hyperplane $T_\alpha \subset \mathfrak{h}^\star$,

$$T_\alpha = \big\{\gamma \in \mathfrak{h}^\star \;\big|\; (\gamma|\alpha) = 0,\ \alpha \in \Phi\big\}, \tag{2.2.95}$$

orthogonal to α. By removing all such hyperplanes we may decompose $\mathfrak{h}^\star$ into connected subsets, called *chambers*. We choose one such subset $\mathcal{C} \subset \mathfrak{h}^\star$ and give it a distinguished

status as the *fundamental chamber*. The fundamental chamber is conventionally chosen as the region enclosed by the hyperplanes T_i orthogonal to the simple roots. This implies that $\mathcal{C}$ contains all vectors $\gamma \in \mathfrak{h}^\star$ such that $(\gamma|\alpha_i)$ is positive,

$$\mathcal{C} = \{\gamma \in \mathfrak{h}^\star \mid (\gamma|\alpha_i) > 0, \forall\ \alpha_i \in \Pi\}, \tag{2.2.96}$$

The chambers in $\mathfrak{h}^\star$ correspond to the images $\omega(\mathcal{C})$ for $\omega \in \mathcal{W}$. The union $\mathcal{X}$ of all such images is called the *Tits cone*,

$$\mathcal{X} = \bigcup_{\omega \in \mathcal{W}} \omega(\mathcal{C}). \tag{2.2.97}$$

For finite-dimensional Lie algebras, the Tits cone coincides with the space $\mathfrak{h}^\star$, while in general, when $\Phi_\Im \neq \emptyset$, one has $\mathcal{X} \neq \mathfrak{h}^\star$.

We are now at a stage where we can describe the root system Φ with more precision. A first important fact is that the sets of real and imaginary roots are separately invariant under the Weyl group,

$$\begin{aligned} \mathcal{W} \cdot \Phi_\Re &= \Phi_\Re, \\ \mathcal{W} \cdot \Phi_\Im &= \Phi_\Im. \end{aligned} \tag{2.2.98}$$

The sets of real and imaginary roots have a decomposition into disjoint sets of positive and negative roots, and we write

$$\begin{aligned} \Phi_\Re &= \Phi_{\Re+} \cup \Phi_{\Re-}, \\ \Phi_\Im &= \Phi_{\Im+} \cup \Phi_{\Im-}, \end{aligned} \tag{2.2.99}$$

with

$$\Phi_{\Re+} \cap \Phi_{\Im+} = 0, \qquad \Phi_{\Re-} \cap \Phi_{\Im-} = 0. \tag{2.2.100}$$

The only root $\alpha \in \Phi_{\Re+}$ for which $s_i(\alpha) \in \Phi_{\Re-}$ is $\alpha = \alpha_i$, implying that

$$s_i \cdot \Phi_{\Re+}/\{\alpha_i\} = \Phi_{\Re+}/\{\alpha_i\}, \tag{2.2.101}$$

and similarly for the negative real roots. A consequence of this is that, since $\alpha_i \notin \Phi_\Im$, the positive and negative imaginary roots are separately invariant under the Weyl group

$$\mathcal{W} \cdot \Phi_{\Im+} = \Phi_{\Im+}, \qquad \mathcal{W} \cdot \Phi_{\Im-} = \Phi_{\Im-}. \tag{2.2.102}$$

We can now state which elements of the root lattice Q are actually roots. First, all real roots lie in Weyl orbits of the simple roots, and thus we have

$$\Phi_\Re = \bigcup_{\omega \in \mathcal{W}} \omega(\Pi). \tag{2.2.103}$$

We now want to find a similar description for the set of imaginary roots. To this end it is useful to first introduce the notion of the *support*, $\mathrm{supp}(\alpha)$, of an element $\alpha \in Q$. Let

$$\alpha = \sum_{i=1}^{r} k_i \alpha_i \ \in\ Q, \tag{2.2.104}$$

and introduce the subdiagram $\Xi_\alpha(A) \subset \Gamma(A)$, of the Dynkin diagram $\Gamma(A)$, as the diagram consisting only of those vertices i for which $k_i \neq 0$ and of all lines joining these vertices. We then have

$$\mathrm{supp}(\alpha) \equiv \Xi_\alpha(A). \tag{2.2.105}$$

Next, we define a region $\mathcal{K} \subset \Phi_{\Im+}$ as follows

$$\mathcal{K} = \big\{\alpha \in Q_+ \;\big|\; \langle \alpha, \alpha_i^\vee \rangle \leq 0,\ \forall\, i\ :\ \Xi_\alpha(A) \text{ is connected}\big\}. \tag{2.2.106}$$

The set of positive imaginary roots can now be elegantly described as the union of all images of $\mathcal{K}$ under the Weyl group [15],

$$\Phi_{\Im+} = \bigcup_{\omega \in \mathcal{W}} \omega(\mathcal{K}). \tag{2.2.107}$$

with a similar description of the negative imaginary roots.

Through the aid of the Weyl group $\mathcal{W}(A)$ we have now obtained a complete description of the root system $\Phi(A)$ of any Kac-Moody algebra $\mathfrak{g}(A)$. In subsequent sections we shall discuss in more detail the root systems for the classes of affine and hyperbolic Kac-Moody algebras, for which some simplifications arise.

We have defined the Weyl group as the group of reflections with respect to the simple roots. Through a natural generalization of (2.2.87) one can define reflections $s_\alpha \in \mathcal{W}$ through any real root $\alpha \in \Phi_{\Re}$. These reflections act as follows on $\beta \in \mathfrak{h}^\star$:

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle\, \alpha = \beta - \frac{2(\beta|\alpha)}{\alpha|\alpha)}\alpha. \tag{2.2.108}$$

How are these reflections related to the fundamental reflections $s_i \equiv s_{\alpha_i}$? To answer this question we first note that since α is real we must have $\alpha = \omega(\alpha_i)$ for some $\omega \in \mathcal{W}$ and some $\alpha_i \in \Pi$. Inserting $\alpha = \omega(\alpha_i)$ into (2.2.108) then yields

$$\begin{aligned} s_\alpha(\beta) &= \beta - \frac{2(\beta|\omega(\alpha_i))}{(\omega(\alpha_i)|\omega(\alpha_i))}\omega(\alpha_i) \\ &= \beta - \frac{2(\omega^{-1}(\beta)|\alpha_i)}{(\alpha_i|\alpha_i)}\omega(\alpha_i) \\ &= \beta - \big\langle \omega^{-1}(\beta), \alpha_i^\vee \big\rangle\, \omega(\alpha_i). \end{aligned} \tag{2.2.109}$$

where we made use of the invariance of $(\cdot|\cdot)$ under the Weyl group. We can rewrite (2.2.109) as

$$\begin{aligned} s_\alpha(\beta) &= \omega\big(\omega^{-1}(\beta) - \big\langle \omega^{-1}(\beta), \alpha_i^\vee \big\rangle\, \alpha_i\big), \\ &= \omega \cdot s_i\big(\omega^{-1}(\beta)\big), \end{aligned} \tag{2.2.110}$$

revealing that the generalized reflection s_α corresponds to a conjugation of the fundamental reflection s_i by some element $\omega \in \mathcal{W}$:

$$s_\alpha = \omega s_i \omega^{-1} \;\in\; \mathcal{W}. \tag{2.2.111}$$

2.3 Affine Kac-Moody Algebras

In this section we shall explore affine Kac-Moody algebras in more detail. This class of algebras corresponds to the first step away from the finite-dimensional Lie algebras, and is the only class of infinite-dimensional Kac-Moody algebras which are well understood. We recall that a Kac-Moody algebra $\mathfrak{g}(A)$ is said to be of affine type if the associated Cartan matrix A is positive semi-definite, det $A = 0$, with one zero eigenvalue. Because of the degeneracy of A, the bilinear form as constructed in Section 2.2.4 is degenerate. We shall explain how this problem is circumvented through the inclusion of an additional generator, called the *derivation* d, in the Cartan subalgebra. This new generator ensures that the invariant bilinear form on the full algebra is non-degenerate.

2.3.1 The Center of a Kac-Moody Algebra

The center Z of a Kac-Moody algebra $\mathfrak{g}(A)$ is defined as follows:

$$Z = \{x \in \mathfrak{g}(A) \,\big|\, \forall\, y \in \mathfrak{g}(A) \;:\; [x, y] = 0\}. \tag{2.3.1}$$

It is a general result that $Z \neq 0$ if and only if $\det A = 0$ [15]. This is related to the rank of the matrix A. In previous sections we have treated the Cartan matrix as an $r \times r$ matrix of matrix rank equal to the rank r of the associated Kac-Moody algebra $\mathfrak{g}(A)$. Now we shall be more general and let $A = (A_{ij})_{i,j=1,\dots,r}$ be an $r \times r$ matrix of *matrix rank* n. In this case, $\det A = 0$ and the rank r Kac-Moody algebra $\mathfrak{g}(A)$ has a non-trivial center $Z \neq 0$ of dimension

$$\dim Z = \text{corank } A = r - n. \tag{2.3.2}$$

For affine Kac-Moody algebras the Cartan matrix has only one zero eigenvalue and hence the corank of A is one, implying that the center is one-dimensional and is spanned by the *central element* c,

$$Z = \mathbb{R}c. \tag{2.3.3}$$

A consequence of this is that affine Kac-Moody algebras are *not* simple, since the center forms a non-trivial ideal in $\mathfrak{g}(A)$. The center Z is always contained in the Cartan subalgebra

$$Z \subset \mathfrak{h}, \tag{2.3.4}$$

implying that the central element c must be expressible as a linear combination $\sum_{i=1}^{r} c_i \alpha_i^\vee$ of the simple corrots $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$. To see this, let $v = (v_1, \dots, v_r)^T$ be the non-trivial element of the kernel of the transposed matrix A^T, i.e.,

$$A^T \cdot v = \sum_{j=1}^{r} (A^T)_{ij} v_j = 0. \tag{2.3.5}$$

We then have $c_i = v_i$, i.e.,

$$c = \sum_{i=1}^{r} v_i \alpha_i^\vee \;\in\; Z \subset \mathfrak{h}. \tag{2.3.6}$$

This result follows from the fact that c must commute with all the Chevalley generators $e_i,\; i = 1, \dots, r$, and hence

$$0 = [c, e_i] = \sum_{j=1}^{r} c_j [\alpha_j^\vee, e_i] = \sum_{j=1}^{r} c_j \left\langle \alpha_i, \alpha_j^\vee \right\rangle e_i = \sum_{j=1}^{r} c_j A_{ji} e_i = \Big[\sum_{j=1}^{r} (A^T)_{ij} c_j\Big] e_i, \tag{2.3.7}$$

which is only satisfied when $\sum_{j=1}^{r} (A^T)_{ij} c_j = 0$, and hence $c_j = v_j$ as announced.

2.3.2 The Derived Algebra and the Derivation

It is now time to define what we mean by an affine Kac-Moody algebra. In fact, the algebra constructed from an "affine" Cartan matrix A using the Chevalley-Serre relations is only the *derived* Kac-Moody algebra

$$\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]. \tag{2.3.8}$$

When $\det A \neq 0$ the derived algebra $\mathfrak{g}'$ coincides with the full algebra $\mathfrak{g}$, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. To understand these statements we must introduce the notion of a *derivation* d. The motivation for this is that the complete Kac-Moody algebra must have a well-defined non-degenerate bilinear form, an object which does not exist for the derived algebra $\mathfrak{g}'$ because of the degeneracy of the Cartan matrix. For the following discussion it will be convenient to make a slight relabelling of the simple roots and the simple coroots. This is motivated

by the fact that for any affine Kac-Moody algebra $\mathfrak{g}$ of rank r one may identify a maximal rank $r-1$ finite-dimensional subalgebra $\bar{\mathfrak{g}} \subset \mathfrak{g}$, and view $\mathfrak{g}$ as an *extension* $\mathfrak{g} \equiv \bar{\mathfrak{g}}^+$ of $\bar{\mathfrak{g}}$, where the superscript "+" indicates that the affine Kac-Moody algebra $\mathfrak{g}$ is obtained by adding a single node to the Dynkin diagram of $\bar{\mathfrak{g}}$ in a prescribed way. We shall discuss the general theory of extensions of Lie algebras in Section 2.4.1 but for now this will suffice. To this end we take the $r = k+1$ simple roots of $\mathfrak{g}(A)$ to be $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_k\}$, with $\bar{\Pi} = \{\alpha_1, \dots, \alpha_k\}$ representing the k simple roots of the finite subalgebra $\bar{\mathfrak{g}}$. The root α_0 is called the *affine root*. It is always of the form

$$\alpha_0 = \delta - \theta, \tag{2.3.9}$$

where δ is a null root, $(\delta|\delta) = 0$, and θ is the highest root of the finite subalgebra $\bar{\mathfrak{g}} \subset \mathfrak{g}$.

We now follow Kac [15] and add by hand a generator $d \in \mathfrak{h}$ to the Cartan subalgebra, with the property

$$\langle \alpha_i, d \rangle = \delta_{i0}, \qquad i = 0, 1, \dots, k. \tag{2.3.10}$$

The basis of the Cartan subalgebra $\mathfrak{h}$ is then taken to be $\Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_k^\vee, d\}$, on which a non-degenerate bilinear form now exists with the following properties:

$$(c|\alpha_i^\vee) = 0, \quad (c|c) = 0, \quad (c|d) = 1, \tag{2.3.11}$$

where the non-degeneracy of $(\cdot|\cdot)$ on $\mathfrak{h}$ follows from the non-vanishing scalar product between the derivation d and the central element c. An *affine* Kac-Moody algebra $\mathfrak{g}(A)$ is then defined as the derived algebra $\mathfrak{g}'$ augmented with the derivation d,

$$\mathfrak{g} = \mathfrak{g}' + \mathbb{R}d. \tag{2.3.12}$$

Note that this implies that an affine Kac-Moody algebra of rank r has a Cartan subalgebra of dimension $r+1$.

Let us now make an effort towards understanding the structure of (2.3.12). Because of its definition, (2.3.10), the derivation $d \in \mathfrak{h}$ will never appear on the right hand side of any commutator in the algebra. For example, the commutation relations with the positive Chevalley generators are

$$\begin{aligned} [d, e_a] &= 0, \quad a = 1, \dots, k, \\ [d, e_0] &= e_0, \end{aligned} \tag{2.3.13}$$

and similarly for the f_i's. This implies that the derived algebra $\mathfrak{g}'$ does not contain the derivation d, thus explaining the structure of (2.3.12). In fact, the derivation can be viewed as a "counting operator" which counts the number of times the affine generator e_0, corresponding to the affine root α_0, appears in any commutator.

2.3.3 The Affine Root System

In the case of affine Kac-Moody algebras, the appearance of imaginary roots does imply a drastic complication. As we have alluded to before, the only "independent" imaginary root is the null root δ, of which all other imaginary roots are multiples. The complete root system Φ is therefore determined by the finite root system $\bar{\Phi}$ of the maximal finite subalgebra $\bar{\mathfrak{g}} \subset \mathfrak{g}$ and the null root δ. As mentioned in the previous section, the Dynkin diagram of $\bar{\mathfrak{g}}$ is obtained by deleting the zeroth node in the Dynkin diagram of the affine algebra $\mathfrak{g}$.

Let A be the Cartan matrix of an affine Kac-Moody algebra $\mathfrak{g}$, with associated root system $\Phi = \Phi(A)$. We begin by splitting Φ into its real and imaginary parts,

$$\Phi = \Phi_{\Re} \cup \Phi_{\Im}. \tag{2.3.14}$$

In the example in Section 2.2.5 we saw that the real roots of A_1^+ were given by all roots of the form $\alpha_1 + n\delta$, $n\mathbb{Z}$, with α_1 being the simple root (and, in fact, the only positive root) of the underlying finite algebra A_1. This fact generalizes to any affine Kac-Moody algebra $\mathfrak{g}(A)$ in the following way

$$\Phi_{\Re} = \big\{\alpha + n\delta \;\big|\; \forall\ \alpha \in \bar{\Phi};\ n \in \mathbb{Z}\big\}, \tag{2.3.15}$$

with $\bar{\Phi}$ being, as usual, the root system of the underlying finite-dimensional Lie algebra $\bar{\mathfrak{g}}$. The positive part of the real roots can then be described as follows

$$\Phi_{\Re+} = \big\{\alpha + n\delta \;\big|\; \forall\ \alpha \in \bar{\Phi};\ n \in \mathbb{Z}_{\geq 0}\big\} \cup \bar{\Phi}_+. \tag{2.3.16}$$

It is well known in Lie theory that if α is a root of a finite-dimensional Lie algebra, then the only multiples of α which are also roots are $\pm\alpha$. This feature carries over to the real part of the root system of general Kac-Moody algebras, while it is no longer true for the imaginary roots. For affine Kac-Moody algebras there exists only one independent imaginary root, and this is the root δ which appears in the construction of the zeroth simple root $\alpha_0 = \delta - \theta$. Any multiple of δ is also an imaginary root, and hence the imaginary part of the root system of any affine Kac-Moody algebra is very easy to describe:

$$\Phi_{\Im} = \Phi_{\Im+} \cup \Phi_{\Im-} = \big\{n\delta \;\big|\; n \in \mathbb{Z}_{\geq 0}\big\} \cup \big\{n\delta \;\big|\; n \in \mathbb{Z}_{\leq 0}\big\}. \tag{2.3.17}$$

2.3.4 The Affine Weyl Group

We now want to perform a closer analysis of the Weyl group of an affine Kac-Moody algebra. During our study we will find a natural explanation for where the name *affine* has its origin.

The Weyl group $\mathcal{W}(A)$ associated with an affine Kac-Moody algebra $\mathfrak{g}(A)$ is defined through the geometric action of the fundamental reflections $\mathcal{S} = \{s_0, s_1, \ldots, s_k\}$ on the dual space $\mathfrak{h}^\star$. Moreover we have seen that one can associate a reflection s_α with respect to any real root $\alpha \in \Phi_{\Re}$ as $s_\alpha = \omega s_i \omega^{-1}$ for $\omega \in \mathcal{W}$ and $\alpha = \omega(\alpha_i)$. On the other hand, no such construction exists for the imaginary roots since for $\beta \in \Phi_{\Im}$ the pairing $\langle \alpha, \beta^\vee \rangle$ is not defined. Therefore, although they act *on* $\Phi_{\Im}$, all Weyl reflections are defined *with respect to* real roots only.

An important new feature owing to the existence of null roots is that, since $(\delta|\alpha) = 0$ for all $\alpha \in \Phi_{\Re}$, we have

$$s_\alpha(\delta) = \delta - \frac{2(\delta|\alpha)}{(\alpha|\alpha)}\alpha = \delta. \tag{2.3.18}$$

This implies that the entire Weyl group acts as the identity on the set of imaginary roots:

$$\omega(\beta) = \beta, \quad \forall\ \beta\ \in\ \Phi_{\Im}. \tag{2.3.19}$$

Note that this is true only in the affine case for which all roots in $\Phi_{\Im}$ are lightlike, but not in the general case when $\Phi_{\Im}$ also contains timelike roots.

Let $\bar{\mathcal{W}} \subset \mathcal{W}$ be the finite Weyl group of $\bar{\mathfrak{g}} \subset \mathfrak{g}$. A particular feature of affine Weyl groups is the fact that they decompose into a semidirect product of the form

$$\mathcal{W} = \bar{\mathcal{W}} \ltimes \bar{T}^\vee, \tag{2.3.20}$$

where $\bar{T}^\vee$ denotes the abelian group of translations of the coroot lattice $\bar{Q}^\vee$ of $\bar{\mathfrak{g}}$.[2] We shall explain this phenomenon in detail below in the context of a simple example.

[2]Here we refer to the coroot lattice $Q^\vee$ as a lattice in $\mathfrak{h}^\star$, in the sense that we can use the non-degenerate bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ to identify $\mathfrak{h}$ with $\mathfrak{h}^\star$. Then the coroot lattice is spanned by the simple coroots $\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i|\alpha_i) \in \mathfrak{h}^\star$ and we have $Q \subset Q^\vee$.

Example: The Weyl group of A_1^+

Let $\Pi = \{\alpha_0, \alpha_1\}$ be the simple roots of $\mathfrak{g} = A_1^+$ and $\bar{\Pi} = \{\alpha_1\}$ the simple root of the underlying finite subalgebra $\bar{\mathfrak{g}} = A_1$. The Weyl group $\bar{\mathcal{W}}$ of A_1^+ is generated by a single reflection s_1 with respect to α_1:

$$s_1(\alpha_1) = -\alpha_1, \tag{2.3.21}$$

implying that $\bar{\mathcal{W}} = \mathbb{Z}_2$. The Cartan matrix of A_1^+ is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

which has a kernel spanned by the column vector $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$. Thus, from our discussion of the center of Kac-Moody algebras in Section 2.3.1, we find that A_1^+ has a central element given by

$$c = \alpha_0^\vee + \alpha_1^\vee. \tag{2.3.22}$$

It is easy to check that c indeed commutes with all generators of the algebra. For example, we have

$$[c, e_1] = [\alpha_0^\vee, e_1] + [\alpha_1^\vee, e_1] = (-2+2)e_1 = 0. \tag{2.3.23}$$

Recall now from Section 2.3.2 that so far we have been dealing only with the derived algebra

$$\mathfrak{g}' = [A_1^+, A_1^+] \tag{2.3.24}$$

whose Cartan subalgebra is

$$\mathfrak{h}' = \mathbb{R}\alpha_0^\vee + \mathbb{R}\alpha_1^\vee. \tag{2.3.25}$$

However, to understand the Weyl group $\bar{\mathcal{W}}$ it is crucial that we treat the full algebra

$$A_1^+ = [A_1^+, A_1^+] + \mathbb{R}d. \tag{2.3.26}$$

Thus, we add by hand the derivation d to the algebra, with the properties

$$(c|d) = 1, \qquad (d|\alpha_0^\vee) = 0, \qquad (d|d) = 0. \tag{2.3.27}$$

In the following we shall also view the central element c as a basis element of the Cartan subalgebra, instead of the generator $\alpha_0^\vee$. This will prove convenient later on. The full Cartan subalgebra now takes the form

$$\mathfrak{h} = \bar{\mathfrak{h}} \oplus \big(\mathbb{R}c + \mathbb{R}d\big), \tag{2.3.28}$$

where $\bar{\mathfrak{h}} = \mathbb{R}\alpha_1^\vee$ is the Cartan subalgebra of A_1.

We shall now proceed to write the simple roots as the root vectors $\vec{\alpha}_0$ and $\vec{\alpha}_1$ with components given by the eigenvalues under the adjoint action of $\mathfrak{h}$ on the Chevalley generators e_0 and e_1. For e_0 we find

$$\begin{aligned} &[\alpha_1^\vee, e_0] = \langle \alpha_0, c\rangle\, e_0 = -2e_0, \\ &[c, e_0] = 0, \\ &[d, e_0] = \langle \alpha_0, d\rangle\, e_0 = e_0, \end{aligned} \tag{2.3.29}$$

and for e_1 we have

$$\begin{aligned} &[\alpha_1^\vee, e_1] = \langle \alpha_1, \alpha_1^\vee\rangle\, e_1 = 2e_1, \\ &[c, e_1] = 0, \\ &[d, e_1] = \langle \alpha_1, d\rangle\, e_1 = 0, \end{aligned} \tag{2.3.30}$$

where we made use of the defining relation, (2.3.10), for the derivation. Consequently, the component forms of the simple root vectors, in the basis determined by $\alpha_1^\vee, c$ and d, are

$$\begin{aligned}\vec{\alpha}_0 &= (-2,0,1),\\ \vec{\alpha}_1 &= (2,0,0).\end{aligned} \tag{2.3.31}$$

We now have all the ingredients to understand the Weyl group of A_1^+. The group $\mathcal{W}$ is generated by the two fundamental reflections s_0 and s_1. We shall compute the action of these generators on an arbitrary vector

$$\vec{\lambda} = (\bar{\lambda}, k, m) \;\in\; \mathfrak{h}^\star. \tag{2.3.32}$$

We begin by computing the action of s_1:

$$\begin{aligned}s_1(\vec{\lambda}) &= \vec{\lambda} - \frac{2(\vec{\lambda}|\vec{\alpha}_1)}{(\vec{\alpha}_1|\vec{\alpha}_1)}\vec{\alpha}_1\\ &= (\bar{\lambda},k,m) - (2\bar{\lambda},0,0)\\ &= (-\bar{\lambda},k,m).\end{aligned} \tag{2.3.33}$$

This result was expected, namely that the action of s_1 on the root lattice $\bar{Q} = \mathbb{Z}\alpha_1$ is the same as for the Weyl group of A_1, i.e., in component form we have

$$s_1(\bar{\lambda}) = -\bar{\lambda}, \qquad s_1(k) = k, \qquad s_1(m) = m. \tag{2.3.34}$$

Now, let us move on to the more interesting case of the s_0-generator. Its action on $\vec{\lambda}$ becomes

$$\begin{aligned}s_0(\vec{\lambda}) &= \vec{\lambda} - \frac{2(\vec{\lambda}|\vec{\alpha}_0)}{(\vec{\alpha}_0|\vec{\alpha}_0)}\vec{\alpha}_0\\ &= (\bar{\lambda},k,m) - \big[-\bar{\lambda}+k\big](-2,0,1)\\ &= (-\bar{\lambda}+2k,k,m+\bar{\lambda}-k).\end{aligned} \tag{2.3.35}$$

We shall focus on the "projected" action of s_0 on the root lattice $\bar{Q}$, which reads

$$s_0(\bar{\lambda}) = -\bar{\lambda} + 2k. \tag{2.3.36}$$

This corresponds to a reflection of $\bar{\lambda}$, not with respect to the origin, but rather with respect to the point displaced by k away from the origin in $\bar{Q}$. We can make things more clear by noting that the combined reflection $t \equiv s_0 \circ s_1$ acts on $\bar{Q}$ as a pure translation:

$$t(\bar{\lambda}) = s_0 \circ s_1(\bar{\lambda}) = \bar{\lambda} + 2k. \tag{2.3.37}$$

Thus the Weyl group of A_1^+, acting on the Euclidean root lattice $\bar{Q}$ of $\bar{\mathfrak{g}}$, contains translations, i.e., *affine transformations*, thus explaining the origin of the name *affine* Kac-Moody algebras.

The action of s_0 can now be written as

$$s_0(\bar{\lambda}) = t \circ s_1(\bar{\lambda}), \tag{2.3.38}$$

which corresponds to the combination of an element $s_1 \in \bar{\mathcal{W}}$ and an element $t \in \bar{T}$, with $\bar{T}$ being the abelian group of translations of the root lattice $\bar{Q}$. This group is generated by t with a general element $t^n \in \bar{T}$ acting as follows

$$t^n(\bar{\lambda}) = \bar{\lambda} + 2kn. \tag{2.3.39}$$

The complete Weyl group of A_1^+ is therefore

$$\mathcal{W} = \big\{t^n,\; t^n \circ s_1 \;\big|\; n \in \mathbb{Z};\; s_1 \in \bar{\mathcal{W}}\big\}, \tag{2.3.40}$$

which is equivalent to the semidirect product

$$\mathcal{W} = \bar{\mathcal{W}} \ltimes \bar{T}, \tag{2.3.41}$$

as announced in the beginning of this section. The reason why it is the root lattice $\bar{Q}$ which appears here, and not the coroot lattice $\bar{Q}^{\vee}$, is that for A_1^+ they coincide, since all roots have the same length. However, in the general case it is the coroot lattice which is relevant.

2.4 Lorentzian Kac-Moody Algebras

Kac-Moody algebras of indefinite type constitute a vast wasteland of unexplored territory. Not much is known in general about these algebras; most notably there is no indefinite Kac-Moody algebra for which the root multiplicities are known in closed form to arbitrary height. We shall here focus on a particular subclass of indefinite Kac-Moody algebras, namely those for which the invariant bilinear form is of Lorentzian signature. These Lorentzian Kac-Moody algebras similarly constitute an infinite class of unclassified algebras, but they nevertheless has a subclass which is, in a certain sense, under control. We refer here to the Lorentzian algebras which can be obtained through prescribed extensions of finite-dimensional simple Lie algebras [18]. Extensions of finite Lie algebras can in this way give rise to Lorentzian Kac-Moody algebras, with properties which have proven to be of interest in string and M-theory. The precise extension procedure is described in detail in Section 2.4.1. Then, in Section 2.4.2, analyze a very interesting subclass of the Lorentzian Kac-Moody algebras, known as *hyperbolic*. As was the case with affine Kac-Moody algebras, we shall see that the hyperbolic class also draws its name from properties of the associated Weyl groups. The hyperbolic Kac-Moody algebras are the only examples of indefinite Kac-Moody algebras which have been completely classified.

2.4.1 Extensions of Lie Algebras

Let A be a symmetrizable generalized Cartan matrix and S its symmetric part. If the bilinear form $(\cdot|\cdot)$ defined by S is of Lorentzian signature then the Kac-Moody algebra $\mathfrak{g}(A)$ is of Lorentzian type. We shall in this section describe how algebras of Lorentzian type can be obtained from finite simple Lie algebras $\bar{\mathfrak{g}} = \mathfrak{g}(\bar{A})$ through double and triple extensions, denoted $\bar{\mathfrak{g}}^{++}$ and $\bar{\mathfrak{g}}^{+++}$, respectively, of the Dynkin diagram $\Gamma(\bar{A})$ associated with $\bar{\mathfrak{g}}$.

As in Section 2.3 we let $\bar{\Pi} = \{\alpha_1, \dots, \alpha_k\}$ be a basis of simple roots for the finite rank k Lie algebra $\bar{\mathfrak{g}}$. The root lattice $\bar{Q} = \sum_{i=1}^{k} \mathbb{Z}\alpha_i$ is Euclidean. Let also $\mathbb{Z}^{1,1}$ be the (unique) even two-dimensional unimodular Lorentzian lattice, spanned by the vectors u_1 and u_2. We define a non-degenerate bilinear form $(\cdot|\cdot)$ on $\mathbb{Z}^{1,1}$, of signature $(-+)$, by

$$(u_1|u_2) = 1, \qquad (u_1|u_1) = 0, \qquad (u_2|u_2) = 0. \tag{2.4.1}$$

This scalar product is induced from the standard Minkowski metric on $\mathbb{R}^{1,1}$ by taking

$$u_1 = (1, 0), \qquad u_2 = (0, -1), \tag{2.4.2}$$

and we write

$$\mathbb{R}^{1,1} = \mathbb{R}u_1 + \mathbb{R}u_2, \qquad \mathbb{Z}^{1,1} = \mathbb{Z}u_1 + \mathbb{Z}u_2. \tag{2.4.3}$$

We now want to extend the Dynkin diagram $\bar{\Gamma} = \Gamma(\bar{A})$ with one node in such a way that the new diagram $\bar{\Gamma}^+$ corresponds to the Dynkin diagram of an affine Kac-Moody algebra $\bar{\mathfrak{g}}^+$. This step was actually already discussed in Section 2.3, and we recall that an affine algebra can be obtained from any finite Lie algebra $\bar{\mathfrak{g}}$ by augmenting the set of simple roots with the *affine root*

$$\alpha_0 \equiv u_1 - \theta, \tag{2.4.4}$$

where θ is the highest root of $\bar{\mathfrak{g}}$ and $u_1 \in \mathbb{Z}^{1,1}$ corresponds to the null root δ. We now have

$$(\alpha_i|u_1) = 0, \quad \text{for} \quad i = 1, \ldots, k, \tag{2.4.5}$$

implying that the new root lattice

$$\bar{Q}^+ = \mathbb{Z}\alpha_0 + \bar{Q} \tag{2.4.6}$$

is contained in the direct sum of $\bar{Q}$ and $\mathbb{Z}^{1,1}$,

$$\bar{Q}^+ \subset \bar{Q} \oplus \mathbb{Z}^{1,1}. \tag{2.4.7}$$

We define new indices $A, B = (0, i)$, such that we can write the matrix of scalar products between the new simple roots $\bar{\Pi}^+ = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ as follows

$$\bar{A}^+_{AB} = \frac{2(\alpha_A|\alpha_B)}{(\alpha_A|\alpha_A)}, \tag{2.4.8}$$

which then corresponds to the entries of the Cartan matrix $\bar{A}^+$ of the affine Kac-Moody algebra $\bar{\mathfrak{g}}^+ = \mathfrak{g}(\bar{A}^+)$.

Let us now proceed to include also the second basis vector $u_2 \in \mathbb{Z}^{1,1}$. This is done by adding the simple root

$$\alpha_{-1} \equiv -u_1 - u_2, \tag{2.4.9}$$

which has non-vanishing scalar product only with α_0:

$$(\alpha_{-1}|\alpha_0) = -1, \qquad (\alpha_{-1}|\alpha_i) = 0, \quad \forall\, i = 1, \ldots, k. \tag{2.4.10}$$

Since we also have $(\alpha_{-1}|\alpha_{-1}) = (\alpha_0|\alpha_0) = 2$ this implies that the node associated with α_{-1} attaches by a single link to the zeroth node of the Dynkin diagram $\bar{\Gamma}^+$ of $\bar{\mathfrak{g}}^+$. As before, we define new collective indices $I, J = (-1, 0, i)$, and the matrix of scalar products

$$\bar{A}^{++}_{IJ} = \frac{2(\alpha_I|\alpha_J)}{(\alpha_I|\alpha_I)} \tag{2.4.11}$$

is the Cartan matrix of the *Lorentzian* Kac-Moody algebra $\bar{\mathfrak{g}}^{++}$. The root lattice $\bar{Q}^{++}$ is then of Lorentzian signature and is equivalent to the direct sum of $\bar{Q}$ with $\mathbb{Z}^{1,1}$,

$$\bar{Q}^{++} = \bar{Q} \oplus \mathbb{Z}^{1,1}. \tag{2.4.12}$$

In order to obtain a triple extension, while still keeping the Lorentzian signature of the root lattice, we introduce yet another two-dimensional Lorentzian lattice $\tilde{\mathbb{Z}}^{1,1}$, spanned by the basis vectors v_1 and v_2. The scalar product on $\tilde{\mathbb{Z}}^{1,1}$ is of the same form as the one on $\mathbb{Z}^{1,1}$,

$$(v_1|v_2) = 1, \qquad (v_1|v_1) = 0, \qquad (v_2|v_2) = 0. \tag{2.4.13}$$

We now note that the vector $v_1 + v_2 \in \tilde{\mathbb{Z}}^{1,1}$ is spacelike, $(v_1 + v_2|v_1 + v_2) = 2$. Thus, by including this vector into the new root lattice we ensure that the Lorentzian signature is preserved, i.e., we do not introduce zero eigenvalues in the bilinear form. We augment the set of simple roots with the "triple-extended" root

$$\alpha_{-2} \equiv u_1 - (v_1 + v_2). \tag{2.4.14}$$

Again, this root is spacelike, $(\alpha_{-2}|\alpha_{-2}) = 2$, and the associated node in the Dynkin diagram connects with a single link to the node corresponding to α_{-1},

$$(\alpha_{-2}|\alpha_{-1}) = -1, \qquad (\alpha_{-2}|\alpha_A) = 0, \quad \forall\, A = 0, 1, \ldots, k. \tag{2.4.15}$$

The new Lorentzian root lattice is given by

$$\bar{Q}^{+++} = \mathbb{Z}\alpha_{-2} + \mathbb{Z}\alpha_{-1} + \mathbb{Z}\alpha_0 + \bar{Q}, \tag{2.4.16}$$

and we have

$$\bar{Q}^{+++} \ \subset\ \tilde{\mathbb{Z}}^{1,1} \oplus \mathbb{Z}^{1,1} \oplus \bar{Q}. \tag{2.4.17}$$

Introducing new indices $M, N = (-2, -1, 0, i)$ we can once again organize the scalar products as

$$\bar{A}^{+++}_{MN} = \frac{2(\alpha_M|\alpha_N)}{(\alpha_M|\alpha_M)}, \tag{2.4.18}$$

corresponding to the Cartan matrix $\bar{A}^{+++}$ of a Lorentzian Kac-Moody algebra $\bar{\mathfrak{g}}^{+++} = \mathfrak{g}(\bar{A}^{+++})$.

Nice examples of extended Lie algebras, which shall be discussed extensively later on, are the Kac-Moody algebras obtained by extending the largest exceptional Lie algebra E_8. This gives rise to the following chain of embeddings

$$E_8 \ \subset E_9 = E_8^{+} \ \subset\ E_{10} = E_8^{++} \ \subset\ E_{11} = E_8^{+++}, \tag{2.4.19}$$

of which E_9 is affine, E_{10} is hyperbolic and E_{11} is Lorentzian (but not hyperbolic). In the next section we shall focus on the subclass of hyperbolic Kac-Moody algebras.

2.4.2 Hyperbolic Kac-Moody Algebras

A hyperbolic Kac-Moody algebra is defined as a Lorentzian algebra which upon removal of any node in the Dynkin diagram yields only finite or (at most one) affine subdiagrams. By this criterion it is easy to understand why E_{11} is not hyperbolic; removal of the node associated with the triple-extended root α_{-2} gives the Dynkin diagram of E_{10} which is neither finite nor affine. In this section we shall see that the class of hyperbolic algebras exhibit some very intriguing features which are unique among all indefinite Kac-Moody algebras.

In the rest of this section we shall take $\mathfrak{g}(A)$ to be a rank r hyperbolic Kac-Moody algebra, unless otherwise specified. By virtue of its Lorentzian signature the space $\mathfrak{h}^\star$ is isomorphic to r-dimensional Minkowski space:

$$\mathfrak{h}^\star \simeq \mathbb{R}^{1,r-1}. \tag{2.4.20}$$

An important consequence of this is that there exists a lightcone in $\mathfrak{h}^\star$, defined as

$$\mathcal{O} = \big\{x \in \mathfrak{h}^\star \ \big|\ (x|x) \leq 0\big\}. \tag{2.4.21}$$

The lightcone clearly separates real and imaginary roots

$$\Phi_{\Im} = \Phi \cap \mathcal{O}. \tag{2.4.22}$$

These properties are of course shared among all Lorentzian algebras. However, a unique feature of hyperbolic Kac-Moody algebras is that its root system is in principle known. By this we mean that any element of the root lattice $Q(A)$ is also an element of the root system $\Pi(A)$ if its norm is less than or equal to 2. In this way we find the following description of the root system:

$$\Phi = \big\{\alpha \in Q \ \big|\ (\alpha|\alpha) \leq 2\big\}. \tag{2.4.23}$$

Furthermore, we recall that the fundamental Weyl chamber $\mathcal{C}$ is defined as the region of $\mathfrak{h}^\star$ which is bounded by the hyperplanes T_i which are orthogonal to the simple roots α_i. As a consequence of their definition, hyperbolic Kac-Moody algebras have the special property

that all these hyperplanes intersect *inside or on the lightcone*. Thus, for any hyperbolic Kac-Moody algebra the fundamental Weyl chamber is contained inside the lighcone

$$\mathcal{C} \subset \mathcal{O}. \tag{2.4.24}$$

Because of the Lorentzian signature, the lightcone decomposes into future and past components, $\mathcal{O}_+$ and $\mathcal{O}_-$, respectively. We shall employ the convention that the simple roots have future temporal directions, implying that the fundamental Weyl chamber, defined as $\{\beta \in \mathfrak{h}^\star \mid (\beta|\alpha_i) > 0,\ i = 1, \dots, r\}$, is actually contained in the *past* lightcone. Since no simple roots are inside the lightcone, the union of all the images of the Weyl group, $\mathcal{W}$, acting on $\mathcal{C}$, will not extend outside of $\mathcal{C}$, and, in fact, we have that the Tits cone, $\mathcal{X}$, coincides with the past lighcone:

$$\mathcal{O} \equiv \mathcal{X} = \bigcup_{\omega \in \mathcal{W}} \omega(\mathcal{C}). \tag{2.4.25}$$

Because of this, the Weyl chamber $\mathcal{C}$ is not a fundamental domain for the action of $\mathcal{W}$ on all of $\mathfrak{h}^\star$, as is the case for finite Lie algebras, but rather is the fundamental domain for the action of $\mathcal{W}$ on the Tits cone, $\mathcal{X}$.

The Weyl group $\mathcal{W}(A)$ of a hyperbolic Kac-Moody algebra $\mathfrak{g}(A)$ is a discrete subgroup of the isometry group of $\mathfrak{h}^\star = \mathbb{R}^{1,r-1}$. Moreover, since all of the hyperplanes T_i are either timelike or lightlike the Weyl group preserves the temporal direction of any $x \in \mathfrak{h}^\star$. This implies that the Weyl group of a hyperbolic Kac-Moody algebra is a subgroup of the ortochronous Lorentz group, $O^\dagger(1, r-1)$, i.e., the time-preserving part of the isometry group of $\mathbb{R}^{1,r-1}$,

$$\mathcal{W} \subset O^\dagger(1, r-1). \tag{2.4.26}$$

Because of this fact, the Weyl group preserves spaces of constant negative curvature in $\mathcal{O}$, i.e., the r-dimensional hyperbolic space $\mathcal{H}_r$. The hyperplanes T_i project onto hyperplanes in $\mathcal{H}_r$ and because we then have $r+1$ hyperplanes bounding a region in an r-dimensional space, the Weyl chamber $\mathcal{C}$ projects onto a simplex of finite volume in $\mathcal{H}_r$. Geometric reflections in the faces of a finite volume simplex in hyperbolic space are elements of a *hyperbolic Coxeter group*. The associated Weyl groups therefore correspond to such hyperbolic Coxeter groups, and it is this fact which is the origin of the name hyperbolic Kac-Moody algebras.

EXAMPLE: THE WEYL GROUP OF A_1^{++}

We have previously discussed the Weyl groups of $\bar{\mathfrak{g}} = A_1$ and $\bar{\mathfrak{g}}^+ = A_1^+$ in some detail. Recall that we found:

$$\mathcal{W}(\bar{A}) = \mathbb{Z}_2, \qquad \mathcal{W}(\bar{A}^+) = \mathbb{Z}_2 \ltimes \bar{T}, \tag{2.4.27}$$

where $\bar{T}$ is the abelian group of translations of the Euclidean root lattice $\bar{Q}$. The first group $\mathbb{Z}_2$ is of course of finite order, while the second one, $\mathbb{Z}_2 \ltimes \bar{T}$, is of infinite order due to the presence of the translation group. Finite Coxeter groups, such as $\mathbb{Z}_2$, are often called *spherical* because they are reflections about the origin in a Euclidean space and so leaves invariant a sphere at infinity. Similarly the affine Coxeter groups leave the entire Euclidean spaces themselves invariant. Finally, on the other side we have the *hyperbolic* Coxeter groups which leave the hyperbolic space invariant. In this section we shall check this more explicitly by investigating the Weyl group of the hyperbolic Kac-Moody algebra $\bar{\mathfrak{g}}^{++} = A_1^{++}$, i.e., the double extension of A_1. The Cartan matrix of this algebra is

$$\bar{A}^{++} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}, \tag{2.4.28}$$

and the associated Dynkin diagram $\Gamma(\bar{A}^{++})$ is displayed in Figure 2.3. We associate a

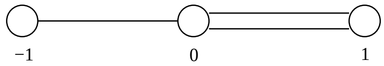

Figure 2.3: The Dynkin diagram of the hyperbolic Kac-Moody algebra A_1^{++}.

fundamental reflection s_I with each of the simple roots $\alpha_I,\ I=-1,0,1$. By making use of Table 2.1 and (2.2.84) we find that these generators obey

$$(s_1 s_0)^{\infty}=1, \qquad (s_0 s_{-1})^3=1, \qquad (s_1 s_{-1})^2=1. \tag{2.4.29}$$

The root space of A_1^{++} is three-dimensional and thus the (projected) Weyl chamber $\mathcal{C}$ is a simplex in the two-dimensional hyperbolic plane $\mathcal{H}_2=\{\beta\in\mathfrak{h}^\star \mid (\beta|\beta)=-1\}$, bounded by the three hyperplanes T_{-1},T_0 and T_1. The angle ϑ_{IJ} between two adjacent adjacent faces I and J are related to the Coxeter exponents, m_{IJ}, as follows (see, e.g., [16])

$$\vartheta=\frac{\pi}{m_{IJ}}. \tag{2.4.30}$$

In our case we thus find that the angle between T_1 and T_0 is zero, the one between T_1 and T_{-1} is $\pi/2$ and the one between T_0 and T_{-1} is $\pi/3$. This implies that the hyperplanes T_1 and T_0 intersect on the border of the lighcone, while the other walls intersect inside the lightcone. This verifies that A_1^{++} is indeed hyperbolic.

We now proceed to show that the Weyl group of A_1^{++} is isomorphic to the extended modular group $PGL(2,\mathbb{Z})$ (see, e.g, [16]). We define the group $PGL(2,\mathbb{Z})$ as the group of 2×2 matrices

$$PGL(2,\mathbb{Z})\ \ni \begin{pmatrix} a & b\\ c & d\end{pmatrix}, \qquad a,b,c,d\ \in\ \mathbb{Z}, \tag{2.4.31}$$

with determinant $ad-bc=1$ and the identification $\{a,b,c,d\}\sim\{-a,-b,-c,-d\}$. [The determinant of any element $X\in GL(2,\mathbb{Z})$ is restricted to $+1$ or -1 in order for the inverse X^{-1} to be contained in the group. In the projected group $PGL(2,\mathbb{Z})$ elements with determinant -1 are projected out.]

Now, we recall that the hyperbolic plane $\mathcal{H}_2$ has a realization as the complex upper half plane $\mathfrak{U}$ (see, e.g, [19])

$$\mathcal{H}_2=\{\tau\in\mathbb{C}\mid \Im(\tau)>0\}. \tag{2.4.32}$$

The group $PGL(2,\mathbb{Z})$ acts on $\tau\in\mathcal{H}_2$ as

$$\tau\ \longrightarrow\ \tau'\ =\ \frac{au+b}{cu+d}, \tag{2.4.33}$$

where we take

$$u=\begin{cases}\tau & ;\quad ad-bc=1\\ \bar\tau & ;\quad ad-bc=-1.\end{cases} \tag{2.4.34}$$

The reason for taking $u=\bar\tau$ when $ad-bc=-1$ is to ensure that the upper half plane is preserved, i.e.,

$$\Im(\tau)\ >\ 0 \quad\Longrightarrow\quad \Im(\tau')\ >\ 0. \tag{2.4.35}$$

We can think of the transformation (2.4.33) as the ordinary action of the modular group $PSL(2,\mathbb{Z})\subset PGL(2,\mathbb{Z})$ together with the action of complex conjugation. Recall that $PSL(2,\mathbb{Z})$ is generated by the translation $\mathcal{T}\ :\ \tau\to\tau+1$ and the inversion $\mathcal{I}\ :\ \tau\to -1/\tau$, with the realization

$$\mathcal{T}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \qquad \mathcal{I}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{2.4.36}$$

The generators of $PGL(2,\mathbb{Z})$ can now be obtained simply by adding the complex conjugation transformation $\tau \longrightarrow -\bar{\tau}$, i.e., we obtain the three generators:

$$\begin{aligned} r_1 \quad &: \quad \tau \longrightarrow -\bar{\tau} \\ r_0 \equiv r_1 \circ \mathcal{T} \quad &: \quad \tau \longrightarrow 1-\bar{\tau} \\ r_{-1} \equiv r_1 \circ \mathcal{I} \quad &: \quad \tau \longrightarrow \frac{1}{\bar{\tau}}. \end{aligned} \tag{2.4.37}$$

These have the matrix realization:

$$r_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad r_0 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad r_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4.38}$$

Using the explicit action of $PGL(2,\mathbb{Z})$ in (2.4.33) one may verify, e.g., that

$$(r_0 r_{-1}) \circ (r_0 r_{-1}) \circ (r_0 r_{-1}) \; : \; \tau \longrightarrow \tau \tag{2.4.39}$$

and similarly that $(r_1 r_{-1})^2 = 1$. We also have that no product of $(r_1 r_0)$ gives the identity, and so we have $(r_1 r_0)^{\infty}$. The group generated by r_1, r_0 and r_{-1} therefore coincides with the Weyl group of A_1^{++} and we may conclude that

$$\mathcal{W}(\bar{A}^{+++}) \;\simeq\; PGL(2,\mathbb{Z}). \tag{2.4.40}$$

Let us finally note that one can see that the groups are the same by comparing the geometric properties of the Weyl chamber with the fundamental domain for the action of $PGL(2,\mathbb{Z})$ on the upper half plane. To this end we write $\tau = x + iy \in \mathfrak{U}$, for $x, y \in \mathbb{R}$, and check that r_1 acts as

$$r_1 \; : \; x + iy \quad \longrightarrow \quad -x + iy, \tag{2.4.41}$$

which implies that this is a reflection in the "hyperplane" $H_1 = \{\tau \in \mathbb{C} \mid \Re(\tau) = 0\}$, i.e., a reflection in the line $x = 0$. By similar arguments one finds that the s_0 transformation is a reflection in the line $x = 1/2$ ($H_0 = \{\tau \in \mathbb{C} \mid \Re(\tau) = 1/2\}$), and that s_{-1} is a reflection in the unit circle $|\tau| = 1$ ($H_{-1} = \{\tau \in \mathbb{C} \mid |\tau| = 1\}$). The angle between H_1 and H_0 is therefore zero, the angle between H_1 and H_{-1} is $\pi/2$ and the angle between H_0 and H_{-1} is $\pi/3$. Hence the $PGL(2,\mathbb{Z})$-elements r_1, r_0 and r_{-1} generate a Coxeter group with Coxeter exponents $m_{10} = \infty, m_{1(-1)} = 2$ and $m_{0(-1)} = 3$, which, again, is the same as for the Weyl group of A_1^{++}.

Chapter 3

Kac-Moody Symmetries Through Compactification

The best candidate for the description and unification of all fundamental interactions is M-Theory. Very little is known for sure about M-Theory, but it is thought to encompass all superstring theories, and in its low energy limit it reduces to eleven-dimensional supergravity. Essential elements are lacking in the quest for a unified theory of quantised gravity and matter. In this context, the study of hidden symmetries, exhibited by dimensional reduction, would allow a better understanding of the structure of the unified theory. These hopes have been encouraged by some developments from recent years that certain types of Kac-Moody algebras occur in several D-dimensional theories of gravity suitably coupled to dilatons and matter fields associated to n-forms, whose Lagrangian is

$$\mathcal{L}_D = \sqrt{-g}\left(R - \frac{1}{2}\sum_{u=1}^{q}\partial_M\Phi^u\partial^M\Phi^u - \sum_n \frac{1}{2n!}e^{\sum_u a_n^u\Phi^u}F_{(n)}^2\right). \tag{3.0.1}$$

The possible existence of these extended symmetries motivates a development of a novel formulation of gravitational theories in which the Kac-Moody symmetries are manifestly built in.

In this section, we will explain how hidden Kac-Moody symmetries are exhibited through compactifications of gravitational theories on a torus. First, we will recall briefly the Kaluza-Klein reduction of pure gravity on a circle S^1. Then, we will consider in detail the reduction of gravity on T^2. In these simple examples we introduce the essential aspects of enhanced symmetries, most notably that of scalar coset Lagrangians and nonlinear realisations. After this, we study the reduction of pure gravity and eleven-dimensional supergravity to 3 dimensions. This leads to the enhancement of the symmetries to a certain finite Lie group $\mathcal{G}$. Finally, we will explain how infinite dimensional Kac-Moody algebras characterise pure gravitational theories and eleven-dimensional supergravity through reduction below three dimensions. In the following we shall for simplicity restrict to the bosonic sectors of all supergravity theories.

3.1 Kaluza-Klein Reduction on S^1

Before starting with the reduction of the bosonic part of supergravity theories on a torus, it will be helpful to briefly recall the Kaluza-Klein reduction on a S^1 (see, e.g., [20]).

First we consider the reduction of pure gravity, described by the Einstein-Hilbert Lagrangian in D dimensions

$$\mathcal{L}_D = \sqrt{-\widehat{g}}\,\widehat{R}. \tag{3.1.1}$$

We put hats on the fields to signify that they are D-dimensional objects. Now suppose that we wish to reduce the theory to $D-1$ dimensions, by compactifying the coordinate $x^{D-1} = z$ on a circle. In general, the enhanced symmetries are manifest only in Einstein frame, and therefore we perform the compactification so as to end up with a $D-1$-dimensional theory in Einstein frame, with a standard kinetic term for the "dilatonic" scalar ϕ. This requirement fixes the compactification ansatz to be

$$d\hat{s}^2 = e^{2\gamma_{D-1}\phi} ds^2 + e^{2\beta_{D-1}\phi}(dz + A_\mu dx^\mu)(dz + A_\nu dx^\nu)\,, \tag{3.1.2}$$

where

$$\gamma_{D-1} = \frac{1}{\sqrt{2(D-2)(D-3)}}\,, \qquad \beta_{D-1} = -(D-3)\gamma_{D-1}. \tag{3.1.3}$$

All the fields on the right-hand side of (3.1.2) are independent of z. Note that this ansatz implies that the components of the higher-dimensional metric $\hat{g}_{MN}$ $(M, N = 0, \ldots, D-1)$ are given in terms of the $D-1$ dimensional fields $g_{\mu\nu}$, A_μ, ϕ $(\mu, \nu = 0, \ldots, D-2)$ by

$$\hat{g}_{\mu\nu} = e^{2\gamma_{D-1}\phi} g_{\mu\nu} + e^{2\beta_{D-1}\phi} A_\mu A_\nu\,, \qquad \hat{g}_{\mu z} = e^{2\beta_{D-1}\phi} A_\mu\,, \qquad \hat{g}_{zz} = e^{2\beta_{D-1}\phi}. \tag{3.1.4}$$

The result of the compactification is

$$\mathcal{L}_{D-1} = \sqrt{-\hat{g}}\,\hat{R} = \sqrt{-g}\,\big(R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{4}e^{-2(D-2)\;\gamma_{D-1}\phi}\,F_{\mu\nu}F^{\mu\nu}\big)\,, \tag{3.1.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Thus, the resulting theory corresponds to $D-1$-dimensional gravity coupled to a scalar field ϕ and a Maxwell field A_μ.

Having seen how the Kaluza-Klein reduction of the metric works, we shall now study the reduction of an antisymmetric tensor field strength, from D to $D-1$ dimensions. Suppose we have an n-form field strength $\hat{F}_{(n)} = d\hat{A}_{(n-1)}$ in D dimensions. In terms of indices, it is clear that after reduction on S^1 there will two kinds of $D-1$ dimensional potentials, namely one with $(n-1)$ indices lying in the $D-1$ dimensional spacetime, and the other with $(n-2)$ indices lying in the $D-1$ dimensional spacetime, and one index being in the compact direction z. This is most easily expressed in terms of differential forms. Thus the ansatz for the reduction of the potential $\hat{A}_{(n-1)}$ is

$$\hat{A}_{(n-1)}(x^\mu, z) = A_{(n-1)}(x^\mu) + A_{(n-2)}(x^\mu) \wedge dz\,. \tag{3.1.6}$$

After reduction, we thus find

$$\sqrt{-\hat{g}}\,\frac{1}{2n!}\hat{F}^2_{(n)} = \sqrt{-g}\,\big(\frac{1}{2n!}\,e^{-2(n-1)\gamma_{D-1}\phi}\,F^2_{(n)} + \frac{1}{2(n-1)!}\,e^{2(D-n-1)\gamma_{D-1}\phi}\,F^2_{(n-1)}\big), \tag{3.1.7}$$

where

$$F_{(n-1)} = dA_{(n-2)}, \qquad F_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge A_{(1)}. \tag{3.1.8}$$

3.2 Reduction of Pure Gravity on T^2 and $SL(2,\mathbb{R})$

It is clear that having established the procedure for performing a Kaluza-Klein reduction from D to $D-1$ dimensions on a circle, the process can be generalized to reduction on a succession of k circles. This is equivalent to a compactification from D to $D-k$ dimensions on an k-torus $T^k = S^1 \times \cdots \times S^1$. We shall in this section restrict to the reduction of pure gravity in D dimensions on a 2-torus, T^2, because many of the features that appear in this analysis are rather general and apply to any kind of toroidal reduction [20].

3.2.1 The $SL(2,\mathbb{R})$-Symmetry of the Reduced Theory

Let us consider the Einstein-Hilbert action in D dimensions:

$$S_D = \int d^D x \sqrt{-g_D}\, R_D\,. \tag{3.2.1}$$

The reduction of this action on S^1 under the ansatz (3.1.2) gives (see (3.1.5))

$$S_{D-1} = \int d^{D-1}x \sqrt{-g_{D-1}} \left(R_{D-1} - \frac{1}{2}\,\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{4}\, e^{-2(D-2)\gamma_{D-1}\phi_1}\, F^2_{(2)\,1} \right). \tag{3.2.2}$$

We perform the same procedure and we compactify again another spacetime direction on S^1. As a result, we obtained a theory reduced on $T^2 = S^1 \times S^1$:

$$\begin{aligned} S_{D-2} = & \int d^{D-2}x \sqrt{-g_{D-2}} \\ & \left(R_{D-2} - \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\,\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{4}\sum_{i=1}^{2} e^{\vec{c}_i\cdot\vec{\phi}} F^2_{(2)\,i} - \frac{1}{2} e^{\sqrt{2}\vec{\alpha}\cdot\vec{\phi}}\, F^2_{(1)12} \right), \end{aligned} \tag{3.2.3}$$

with

$$\vec{c}_1 = \left(-\sqrt{\frac{2(D-2)}{(D-3)}}, -\sqrt{\frac{2}{(D-3)(D-4)}} \right), \tag{3.2.4}$$

$$\vec{c}_2 = \left(0, -\sqrt{\frac{2(D-3)}{(D-4)}} \right), \tag{3.2.5}$$

$$\vec{\alpha} = \left(-\sqrt{\frac{(D-2)}{(D-3)}}, \sqrt{\frac{(D-4)}{(D-3)}} \right), \tag{3.2.6}$$

where the indices μ traverse the non compact dimensions and the inferior indices $_{()}$ of $F_{(p)}$ indicate that it is a p-form associated to a $p-1$-form potential $A_{(p-1)}$. The other indices are label indices that specify the torus T^i where the fields appeared. The fields resulting from this reduction are $g_{\mu\nu}$, $A_{(1)i}$, $A_{(0)12}$, and, $\vec{\phi} = (\phi_1, \phi_2)$. The scalar field $A_{(0)12}$ called *axion* comes from the dimensional reduction of the first Kaluza-Klein vector $A_{(1)1}$.

Let us now look at the scalars in the reduced theory (3.2.3) described by the scalar Lagrangian

$$\mathcal{L}_{scalar} = -\frac{1}{2}\,\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}\,\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}\, e^{\sqrt{2}\vec{\alpha}\cdot\vec{\phi}}\,\partial_\mu\chi\,\partial^\mu\chi\,, \tag{3.2.7}$$

where $\chi = A_{(0)12}$. Things simplify a lot if we rotate the basis for the two dilatons $\vec{\phi} = (\phi_1, \phi_2)$. If we make the orthogonal transformations to two new dilaton combinations, which we may call ϕ and φ:

$$\phi = \frac{1}{2}\left(-\sqrt{\frac{2(D-2)}{(D-3)}}\,\phi_1 + \sqrt{\frac{2(D-4)}{(D-3)}}\,\phi_2 \right), \tag{3.2.8}$$

$$\varphi = \frac{1}{2}\left(-\sqrt{\frac{2(D-4)}{(D-3)}}\,\phi_1 - \sqrt{\frac{2(D-2)}{(D-3)}}\,\phi_2 \right), \tag{3.2.9}$$

the Lagrangian (3.2.7) becomes

$$\mathcal{L}_{scalar} = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\, e^{2\phi}\,\partial_\mu\chi\,\partial^\mu\chi\,. \tag{3.2.10}$$

To analyse the global symmetries of this system, we will consider φ independently of ϕ and χ (because φ is decoupled from the others). On the one hand, $\mathcal{L}_{scalar}$ has a global shift symmetry $\varphi \rightarrow \varphi + k$. This gives an $\mathbb{R}$ factor in the global symmetry group. On the other hand, the part of $\mathcal{L}_{scalar}$ containing ϕ and χ is invariant under the transformations of $SL(2,\mathbb{R})$. Indeed, if we define a complex field $\tau = \chi + i\, e^{-\phi}$ the Lagrangian for ϕ and χ can be written as

$$\mathcal{L}_{(\phi,\chi)} \equiv -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,e^{2\phi}\,\partial_\mu\chi\,\partial^\mu\chi = -\frac{\partial_\mu\tau\,\partial^\mu\bar{\tau}}{2\,\tau_2^2}\,, \tag{3.2.11}$$

where τ_2 is the imaginary part of $\tau = \tau_1 + i\tau_2$. Now it is not hard to see that this Lagrangian is invariant under the transformation:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad \text{with } ad - bc = 1\,, \tag{3.2.12}$$

where a, b, c and d are constants $\in \mathbb{R}$. This transformation can be written under matrix form:

$$\mathbf{\Lambda} = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \tag{3.2.13}$$

with the condition $\det \Lambda = 1$. What we have here is real 2×2 matrices of unit determinant. They form the group $SL(2,\mathbb{R})$. This symmetry acts nonlinearly on the complex field τ as in (3.2.12). To conclude, we have seen that the scalar Lagrangian (3.2.10) as in total an $SL(2,\mathbb{R}) \times \mathbb{R}$ global symmetry. This makes a $GL(2,\mathbb{R})$ symmetry.

We can show that the non-scalar part of the reduced theory (3.2.3) also share the same symmetry that the scalar part. Note that while the scalars transform nonlinearly under $SL(2,\mathbb{R})$, the two gauge potentials $A_{(1)1}$ and $A_{(1)2}$ transform linearly, as a doublet. Thus the global symmetry of the lower-dimensional Lagrangian is already established by looking just at the scalar fields, and their symmetry transformations.

3.2.2 Scalar Coset Lagrangian

To understand the structure of the global symmetry better, we need to study the nature of the scalar Lagrangian that arise from the dimensional reduction on the torus T^2. It leads us into the subject of *nonlinear σ-models* and *coset spaces*. The example of $SL(2,\mathbb{R})$ exhibits many of the general features that one encounters in nonlinear σ-models, while having the merit of being rather simple and easy to calculate explicitly.

The associated Lie algebra of $SL(2,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{R}) \simeq A_1$. We have seen in Section 2.1 that his Chevalley basis is $\{e, f, h\}$ (2.1.4).

Consider now the exponentiation of the h and e, and define the *coset representative*:

$$\begin{aligned} \mathcal{V} &= e^{\frac{1}{2}\phi\, h}\, e^{\chi\, e}\,, && (3.2.14) \\ &= \left(\begin{array}{cc} e^{\frac{1}{2}\phi} & \chi\, e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{array} \right), && (3.2.15) \end{aligned}$$

where ϕ and χ are fields depending on the coordinates of a $D-1$-dimensional spacetime. This matrix is in the upper-triangular form and we said that $\mathcal{V}$ is in the upper-triangular gauge or in the *Borel gauge* (we have a Borel gauge because $\mathcal{V}$ is constructed by exponentiation of the Borel subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ defined in Section 2.1). If we define

$$\mathcal{M} = \mathcal{V}^T\mathcal{V}\,, \tag{3.2.16}$$

we can write a Lagrangian as

$$\begin{aligned} \mathcal{L} &= \frac{1}{4}\,tr(\,\partial_\mu\mathcal{M}^{-1}\,\partial^\mu\mathcal{M}\,)\,, && (3.2.17) \\ &= -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,e^{2\phi}\partial_\mu\chi\,\partial^\mu\chi\,. && (3.2.18) \end{aligned}$$

This is exactly the $SL(2,\mathbb{R})$-invariant scalar Lagrangian $\mathcal{L}_{(\phi,\chi)}$ (3.2.11). The coset representative allowed to find a pleasant form for building the scalar Lagrangian using the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. The advantage now is that we have a very nice way to see why it is $SL(2,\mathbb{R})$ invariant. To do this, if we consider a transformation $\Lambda \in SL(2,\mathbb{R})$, then

$$\mathcal{V} \longrightarrow \mathcal{V}'' = \mathcal{V}\Lambda\,, \tag{3.2.19}$$

thus

$$\mathcal{M} \longrightarrow \Lambda^T \mathcal{M}\,\Lambda\,, \tag{3.2.20}$$

which manifestly leaves the Lagrangian (3.2.17) invariant (using the cyclic invariance of the trace). However, we did something improper because the transformation $\mathcal{V} \to \mathcal{V}''$ do not leave $\mathcal{V}''$ in the upper-triangular form that the original matrix $\mathcal{V}$ given in (3.2.14). It is thus necessary to make a compensating local transformation $\mathcal{O}$ (local in the sense that this matrix depends not only on the constant $SL(2,\mathbb{R})$ parameters but also on the fields ϕ and χ) that does the job of restoring $\mathcal{V}''$ to the upper-triangular gauge. The matrix $\mathcal{O}$ acts on $\mathcal{V}''$ from the left, at the same time as we multiply by Λ from the right. We define, thus a transformed matrix $\mathcal{V}'$ by

$$\mathcal{V}' = \mathcal{O}\,\underbrace{\mathcal{V}\Lambda}_{\mathcal{V}''}\,, \tag{3.2.21}$$

where the matrix $\mathcal{O}$ found is orthogonal. We can now interpret the action of $SL(2,\mathbb{R})$ in terms of transformations on the fields ϕ and χ. We can again easily check that the transformation $\mathcal{V}'$ (3.2.21) leaves the Lagrangian (3.2.17) invariant.

At a given spacetime point (i.e. for fixed values of ϕ and χ), we can use the $SL(2,\mathbb{R})$ transformation to get from any pair of values for ϕ and χ, any other pair of values. This means that $SL(2,\mathbb{R})$ acts transitively on the *scalar manifold* which is the manifold where the fields ϕ and χ take their values. But we must make a compensation transformation $O(2)$ to make sure that we stay within our original upper-triangular form. Thus we may specify points in the scalar manifold by the *coset* $SL(2,\mathbb{R})/O(2)$, consisting of $SL(2,\mathbb{R})$ motions modulo the appropriate $O(2)$ compensators.

3.3 Compactification on T^k

Having established the procedure for performing a Kaluza-Klein reduction from D to $D-1$ dimensions on the circle S^1, it is clear that the process can be repeated for a succession of circles. Thus we will consider a reduction from D to $D-k$ dimensions on a k-torus T^k [21]. At each i'th reduction step, one generates a Kaluza-Klein vector potential $A^i_{(1)}$, and a dilaton ϕ_i from the reduction of the metric (see (3.1.5)). In addition, from p-form potential present in $D-i$ dimensions, one generates a p-form and a $p-1$ form potential (see (3.1.7)). As a result, one obtains rapidly a proliferating number of fields by compactification on T^k.

In addition, once we reach $n+1$ dimensions, we can dualise the n-form $F_{(n)}$ (present in the Lagrangian $\mathcal{L}_D$ (3.0.1) into a scalar. We suppose that the action in $n+1$ dimensions has the form

$$S = \int d^{n+1}x\,\frac{\sqrt{-g}}{2n!}\,e^{\vec{\alpha}\cdot\vec{\phi}}\,F^2_{(n)}\,. \tag{3.3.1}$$

We introduce in (3.3.1), the field φ as a Lagrange multiplier term

$$S = \int d^{n+1}x\,\frac{\sqrt{-g}}{2n!}\,e^{\vec{\alpha}\cdot\vec{\phi}}\,F^2_{(n)} + \frac{\sqrt{-g}}{n!}\,\varphi\,\partial_\mu F_{\nu_1\ldots\nu_n}\,\epsilon^{\mu\nu_1\ldots\nu_n}\,. \tag{3.3.2}$$

Variation with respect to φ simply enforces the Bianchi identity on $F_{(n)}$. However we can also integrate by parts so that there are no derivatives acting on $F_{(n)}$. Eliminating $F_{(n)}$ by its algebraic equation of motion

$$F_{\nu_1 \ldots \nu_n} = e^{-\vec{\alpha}\cdot\vec{\phi}}\, \epsilon_{\mu\nu_1 \ldots \nu_n}\, \partial^\mu \varphi\,, \tag{3.3.3}$$

and substituting back into (3.3.2) leads to the equivalent scalar action

$$S = \int d^{n+1}x\, \frac{\sqrt{-g}}{2}\, e^{-\vec{\alpha}\cdot\vec{\phi}}\, \partial_\mu \varphi\, \partial^\mu \varphi\,. \tag{3.3.4}$$

In conclusion, after the reduction of (3.0.1) on the torus T^k, several scalar fields appears:

- The dilatons Φ^u, $u = 1, \ldots q$ present initially in the D dimensional theory,
- k scalars ϕ_i, $i = 1, \ldots k$ obtained from the reduction of the metric,
- scalars from the reduction of the 2-form $F^k_{\mu\nu}$ (associated to the potential A^k_μ) generated at each step by the dimensional reduction of gravity from $D-k+1$ to $D-k$ dimensions,
- scalars resulting from the potentials associated to the $F_{(n)}$ present initially in the Lagrangian $\mathcal{L}_D$ (3.0.1). These scalars appear in the compactification on the torus T^{n-1} for $k \geq n-1$,
- scalars obtained by dualization of all n-form $F_{(n)}$ when $k = D-n-1$. In particular, when $D = 3$ (compactification on T^{D-3}), all the 2-form $F^k_{\mu\nu}$ ($k = 1, \ldots, D-3$), resulting from the reduction of the metric, can be dualize into scalars.

The three last kind of scalars will be denoted by χ_α.

When we perform the reduction of the Lagrangian (3.0.1) to $D = 3$, we have seen that all the fields can be dualised into scalars and the reduced quadratic Lagrangian in $D = 3$ takes the form:

$$\mathcal{L}_3 = \sqrt{-g}\left(R - \frac{1}{2}\,\partial_\mu \vec{\varphi}\cdot\partial^\mu\vec{\varphi} - \frac{1}{2}\sum_\alpha e^{\sqrt{2}\,\vec{\alpha}\cdot\vec{\varphi}}\,\partial_\mu\chi_\alpha\,\partial^\mu\chi_\alpha\right), \tag{3.3.5}$$

where $\vec{\varphi} = \{\Phi^u, \phi_i\} = (\Phi^1, \ldots, \Phi^q; \phi_1, \ldots, \phi_{D-3})$ and where the vectors $\vec{\alpha}$ are constant vectors with $(q + D - 3)$ components, characterised the scalars χ_α.

One expects that the symmetry of this reduced Lagrangian will be $GL(D-3, \mathbb{R})$ which is the symmetry group on the $D-3$-torus. But for some very specific theories, this symmetry is much larger. In fact, under certain conditions, the scalar part of the reduced Lagrangian $\mathcal{L}_3$ (3.3.5) can be identified to a coset Lagrangian $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$ invariant under the transformation $\mathcal{G}/\ \mathcal{K}$ where $\mathcal{G}$ is a simple Lie group and $\mathcal{K}$ is the maximal compact subgroup of $\mathcal{G}$. This identification was made in details in the case of the reduction of gravity in the torus T^2 (see Section 3.2.2).

We will consider now the reduction to $D = 3$ of gravity and the eleven-dimensional supergravity. We will see that upon dimensional reduction down to three dimensions, the pure gravity exhibits the simple Lie group $SL(D-2, \mathbb{R})$ associated to the algebra A_{D-3} and the bosonic part of eleven-dimensional supergravity exhibits the simple Lie group $\mathcal{E}_8$ associated to the Lie algebra E_8. In the following section, we will show how to obtain the Dynkin diagram of A_{D-3} and E_8 by identifying the vectors $\vec{\alpha}$ in the reduced Lagrangian $\mathcal{L}_3$ (3.3.5) with the positive roots of related Lie algebra $\mathfrak{g}$.

3.4 Reduction of Pure Gravity to 3 Dimensions and $SL(D-2,\mathbb{R})$

We will consider in this section the reduction of gravity from D to 3 dimensions and we will show how we obtain the Dynkin diagram of A_{D-3} [21].

There are two kinds of scalars corresponding to de $D-3$ simple roots of $A_{D-3} \simeq \mathfrak{sl}(D-2,\mathbb{R})$.

On the one hand, $D-4$ scalars $g_{D-k,D-k-1}$ ($k = 1,\ldots,D-4$) are resulting from the dimensional reduction from $D-k$ to $D-k-1$ dimensions, of the Kaluza-Klein potential A^k_μ. The related simple roots are vectors of $D-3$ components

$$\vec{\alpha}^g_k = \Bigg(\underbrace{0,\ldots,0}_{k-1\,terms}, -\sqrt{2}\,(D-k-1)\,\gamma_{D-k}, \sqrt{2}\,(D-k-3)\,\gamma_{D-k-1}, \underbrace{0,\ldots,0}_{D-k-4\,terms} \Bigg), \quad (3.4.1)$$

where γ_D given by (3.1.3). These simple roots[1] define the Lie algebra A_{D-4}. It is easy to find:

$$(\alpha^g_k|\alpha^g_l) = \begin{cases} 2 & k = l\,, \\ 1 & |k-l| = 1\,, \\ 0 & |k-l| > 1\,. \end{cases} \quad (3.4.2)$$

The associated Dynkin diagram, displayed in Figure 3.1, contains $D-4$ vertices and defines *the gravity line*[2] where α^g_1 appears on T^2, α^g_2 on T^3, …, and α^g_{D-4} on T^{D-3}. On the

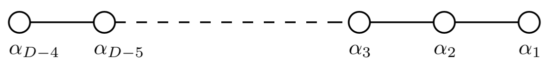

Figure 3.1: Dynkin diagram of A_{D-4}.

other hand, all the Kaluza-Klein potential (A^k_μ) can be dualised into scalars when $D = 3$ using (3.3.4). Only the dualization of the first Kaluza-Klein potential that appears in $D-1$ dimensions in the dimensional reduction of the metric, will be related to a simple root [3]

$$\vec{\alpha}^{gp} = \Big(\sqrt{2}\,(D-2)\,\gamma_{D-1}, \sqrt{2}\,\gamma_{D-2}, \sqrt{2}\,\gamma_{D-3}, \ldots, \sqrt{2}\,\gamma_3 \Big)\,. \quad (3.4.3)$$

This simple root obeys to

$$\begin{aligned} (\alpha^{gp}|\alpha^{gp}) &= 2\,, & (3.4.4) \\ (\alpha^{gp}|\alpha^g_k) &= -\delta_{k,1}\,, & (3.4.5) \end{aligned}$$

where the second relation indicates that α^{gp} is related by a line to the vertex α^g_1 of the gravity line. Thus the Einstein-Hilbert action reduced to 3 dimensions has a coset structure and the associated Dynkin diagram A_{D-3}, displayed in Figure 3.2, corresponding to the group $SL(D-2,\mathbb{R})$. The rank of A_{D-3} is $D-3$ with a total of $\frac{1}{2}(D-2)(D-3)$ positive roots whose $D-3$ are simple. The necessary condition to find this symmetry $SL(D-2,\mathbb{R})$ is that the vectors $\vec{\alpha}$ of the reduced Lagrangian (3.3.5) correspond to all the positive roots of the algebra A_{D-3}.

[1] There exists in addition to these $D-4$ simple roots, $\frac{1}{2}(D-5)(D-4)$ non-simple positive roots related to the other axions. There are non-simple because they can be written as linear combination of α^g_k.

[2] The index $'g'$ of α^g_k will be removed in all Dynkin diagrams for more clearness. The simple roots belonging to the gravity line will be always represented by white vertices.

[3] If we dualize in $D = 3$ the remaining vectors, that leads to $D-4$ non-simple positive roots in the form $\alpha^{gp} + \sum_{l=1}^{p} \alpha^g_l$, with $p = 1,\ldots,D-4$.

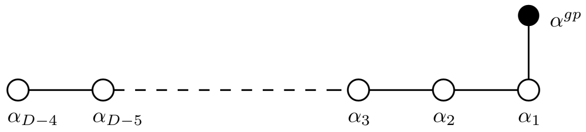

Figure 3.2: Dynkin diagram of A_{D-3}.

3.5 Reduction of Eleven-Dimensional Supergravity to $D = 3$ and $\mathcal{E}_8$

We will consider in this section the reduction to 3 dimensions of eleven-dimensional supergravity [21] whose bosonic action is

$$\mathcal{S}^{(11)} \quad = \quad \frac{1}{16\pi G_{11}} \int d^{11}x\sqrt{-g^{(11)}}\Big(R^{(11)} - \frac{1}{2\cdot 4!}F_{\mu\nu\sigma\tau}F^{\mu\nu\sigma\tau} + C.S.\Big). \qquad (3.5.1)$$

The study of the symmetry of this theory is crucial because it is thought to be the low energy effective action of M-theory.

It is obvious if we reduce this theory on $T^{D-3} = T^8$, that we obtain exactly the same scalar fields and the same associated roots as the ones obtained by the dimensional reduction of pure gravity (see Section 3.4). In addition to α_k^g and α^{gp}, we obtain others roots associated to scalar fields coming from the dimensional reduction of the 4-form $F_{(4)}$ present in the unreduced action (3.5.1). The first scalar resulting from the compactification of $F_{(4)}$ appears on T^3 and the associated vector $\vec{\alpha}_4^E$ can be written as

$$\vec{\alpha}_4^E = \Big(\underbrace{\sqrt{2}\,.6\,\gamma_8,\ldots,\sqrt{2}\,.6\,\gamma_8}_{3\,terms},\underbrace{0,\ldots,0}_{5\,terms}\Big). \qquad (3.5.2)$$

When this vector corresponds to a root, it is called *electric root*[4]. It is easy to see that this electric root satisfy the following relations:

$$(\alpha_4^E|\alpha_{4)}^E \quad = \quad 2\,, \qquad (3.5.3)$$
$$(\alpha_4^E|\alpha_k^g) \quad = \quad -\delta_{k,3}\,, \qquad (3.5.4)$$
$$(\alpha_4^E|\alpha^{gp}) \quad = \quad 1\,. \qquad (3.5.5)$$

The second relations implies that the simple electric root α_4^E is connected to the third vertex of the gravity line α_3^g and the last relations implies that α^{gp} is not any more a simple root.

There is another scalar obtained at 5 dimensions by the dualization of the $F_{(4)}$. The corresponding vector can be written as

$$\vec{\alpha}_4^M = \Big(\underbrace{3\sqrt{2}\,\gamma_{10},3\sqrt{2}\,\gamma_9,\ldots,3\sqrt{2}\,\gamma_5}_{6\,terms},0,0\Big). \qquad (3.5.6)$$

This vector is a root and it is called *magnetic root.* This magnetic root is not simple because it can be written as linear combination of α_k^g and α_4^E.

The set of all simple roots gives the Dynkin diagram of E_8 displayed in Figure 3.3, and the other scalars present in 3 dimensions are related to all non-simple positive roots of E_8.

[4] The other scalars resulting from the compactification of $F_{(4)}$ beyond 8 dimensions, are associated to roots that are linear combination with positive coefficients of α_k^g and α_4^E. They are then non-simple roots.

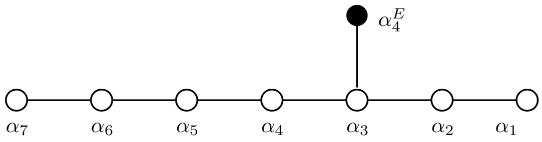

Figure 3.3: Dynkin diagram of E_8.

3.6 The Coset Lagrangian $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$

We have seen that the vectors $\vec{\alpha}$ of $\mathcal{L}_3$ (3.3.5) are identified to positive roots of A_{D-3} in the case of reduction of gravity and to the positive roots of E_8 in the case of the reduction of eleven-dimensional supergravity. The fact that we found that $\vec{\alpha}$ are identified to positive roots of a simple Lie algebra $\mathfrak{g}$ is a necessary condition that the reduced Lagrangian in 3 dimensions containing only scalars is invariant under $\mathcal{G}$ (= group whose the Lie algebra is $\mathfrak{g}$).

In this section, we will show that the scalar part of the reduced Lagrangian $\mathcal{L}_3$ (3.3.5) can be identified to a coset Lagrangian $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$ where $\mathcal{G}$ is a simple Lie group and $\mathcal{K}$ is the maximal subgroup of $\mathcal{G}$. This identification was made in details in the case of the reduction of gravity in the torus T^2 (see Section 3.2).

First, let us take a closer look at the subgroup $\mathcal{K}$. In the case of the compactification of gravity on T^2, we have found that $\mathcal{K} = O(2)$. However, it is clear that performing more compactificactions on T^k it would become increasingly complicated to construct the compensator $\mathcal{O}$. There is fortunately a general theorem in the theory of Lie algebras that claims that the compensator $\mathcal{O}$ is an element of the maximal compact subgroup $\mathcal{K}$ of $\mathcal{G}$. This is the Iwasawa decomposition, which was introduced in Section 2.2.3. Here we introduce the Iwasawa decomposition at the group level. We then have the following statement: *every element g in the Lie group $\mathcal{G}$, associated to a Lie algebra $\mathfrak{g}$, can be uniquely expressed as the following product:*

$$g = g_{\mathfrak{k}}\, g_{\mathfrak{h}}\, g_{\mathfrak{n}_+}\,, \tag{3.6.1}$$

where $g_{\mathfrak{k}}$ belongs to the maximal compact subgroup $\mathcal{K}$, with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$, $g_{\mathfrak{h}}$ belongs to the Cartan torus $\mathfrak{h}$ of $\mathfrak{g}$ and $g_{\mathfrak{n}^+}$ belongs to the nilpotent part of $\mathcal{G}$. Our coset representative $\mathcal{V}$ will be constructed by exponentiating the Cartan generators and the full set of positive-root generators (see (3.2.14)). Thus our coset representative is written as $\mathcal{V} = g_{\mathfrak{h}}\, g_{\mathfrak{n}_+}$. Now, if we act by right-multiplication with a general group element Λ in $\mathcal{G}$, then $\mathcal{V}\Lambda$ is some element of the group $\mathcal{G}$. Now, invoking the Iwasawa decomposition, we must be able to write the group element $\mathcal{V}\Lambda$ in the form $g_{\mathfrak{k}}\mathcal{V}'$ where $\mathcal{V}'$ itself is of the form $g_{\mathfrak{h}'}\, g_{\mathfrak{n}'_+}$. This assures that there exists a way of pulling out an element $\mathcal{O}$ of the maximal subgroup $\mathcal{K}$ of $\mathcal{G}$ on the left-hand side, such that $\mathcal{V}\Lambda = \mathcal{O}\mathcal{V}'$.

To construct the maximal subgroup $\mathcal{K}$ of $\mathcal{G}$, we use the Chevalley involution ω defined in (2.2.41) which has the effect of reversing the sign of every non-compact generator in the algebra $\mathfrak{g}$, while leaving the sign of every compact generator unchanged.

The coset representative is build by exponentiating the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ (see Section 2.2.3)

$$\mathcal{V} = e^{\frac{1}{\sqrt{2}}\,\vec{\varphi}\cdot\vec{h}} \sum_{\alpha>0} e^{\chi_\alpha\, e_\alpha}\,, \tag{3.6.2}$$

where $\vec{h} = \{R_u, h_i\}$. The generators R_u, $u = 1, \ldots, q$ are generators associated to the dilatons Φ^u present in the non-reduced theory in D dimensions. The h_i $(i = 1, \ldots, D-3)$ and R_u form the Cartan subalgebra $\mathfrak{h}$ (defined in Section 2.2.1) of $\mathfrak{g}$ and the generator e_α is the generator associated to the positive root α (see (2.2.21)). We can identify the scalar

part of the reduced Lagrangian (3.3.5) to a coset Lagrangian

$$\mathcal{L}_{\mathcal{G}/\mathcal{K}} = \frac{1}{4}\left(\partial_\mu \mathcal{M}^{-1} | \partial^\mu \mathcal{M}\right), \qquad (3.6.3)$$

$$= \frac{1}{4}\, tr(\partial_\mu \mathcal{M}^{-1}\, \partial^\mu \mathcal{M}), \qquad (3.6.4)$$

where $\mathcal{M} = \mathcal{V}^\sharp\, \mathcal{V}$ and $(\cdot|\cdot)$ is the invariant bilinear form on the Lie algebra (see Section 2.2.4) that corresponds to the trace in the case of finite-dimensional simple Lie algebra. We define the '*generalised transpose* ' $X^\sharp$ on a generator X by

$$X^\sharp \equiv \omega\,(X^{-1}), \qquad (3.6.5)$$

where ω is the Chevalley involution defined in Section 2.2.3. In the simple cases, corresponding to orthogonal subgroups $\mathcal{K}$ (like $O(2)$), $\sharp$ coincides with the transpose. If we normalise in the adjoint representation the Cartan and the positive root generators so that

$$tr(h_i\, h_j) = \delta_{ij}\,, \qquad tr(e_\alpha\, e_\beta) = 0\,, \qquad tr(e_\alpha\, f_\beta) = \delta_{\alpha\beta}\,, \qquad (3.6.6)$$

then one can show that $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$ is precisely the scalar part of (3.3.5). Thus it follows that if the vectors $\vec{\alpha}$ obtained from the compactification can be identified with positive roots of an algebra $\mathfrak{g}$, then the action when dimensionally reduced to three dimensions has a symmetry $\mathcal{G}/\mathcal{K}$. Indeed, we can easily check that $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$ is invariant under the global transformation $\Lambda \in \mathcal{G}$ and under the local transformations $\mathcal{O} \in \mathcal{K}$.

$$\mathcal{V} \longrightarrow \mathcal{O}\,\mathcal{V}\,\Lambda\,. \qquad (3.6.7)$$

If we consider our two examples analysed in Section 3.4 and 3.5, the reduced Lagrangian in 3 dimensions is invariant under transformations of $SL(D-2,\mathbb{R})/O(D-2)$ for the theory containing only gravity and it is invariant under the transformations of $\mathcal{E}_8/SO(16)$ for the eleven-dimensional supergravity.

3.7 Dimensional Reduction Below 3 Dimensions and Kac-Moody Algebras

In the precedent sections, we have performed the dimensional reduction to 3 dimensions of pure gravity and of eleven-dimensional supergravity. That leads to a reduced Lagrangian (3.3.5) containing only scalars coupled to gravity in 3 dimensions. It is interesting to consider the dimensional reduction beyond 3 dimensions[5] but it is not any more possible to reduce these theories on the torus T^k for $k > D-3$. Indeed, the gravity does not exist below 3 dimensions and a later compactification will not give additional scalar fields in the coset Lagrangian $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$. Nevertheless, the symmetries of the systems must increase because one expects that all the theories posses at least a global symmetry $GL(k,\mathbb{R})$ when they are reduced on T^k. Thus it is obvious to extend the gravity line when we perform a compactification beyond 3 dimensions.

The dimensional reduction beyond 3 dimensions would give Dynkin diagrams of Kac-Moody algebras. It has been showed that the reduced theory to 2 dimensions are connected to a infinite dimensional symmetry $\mathcal{G}^+$ (affine extension of $\mathcal{G}$) [22] obtained by adding one root to the Dynkin diagram of $\bar{\mathfrak{g}}$ where $\bar{\mathfrak{g}}$ is a finite Lie algebra (see Section 2.3). The first example of affine 'hidden' symmetries is the affine symmetry group $SL(2,\mathbb{R})^+$, known as the Geroch group. It comes from the study of solutions of the Einstein equations of

[5] The process using previously to reduce the different theories must be renounced because the parameter $\gamma_{D-1} = \frac{1}{\sqrt{2(D-2)(D-3)}}$ present in all the components of the root of algebra $\mathfrak{g}$, is badly defined for $D = 2$ and 3.

pure four-dimensional gravity admitting Killing vectors. The Ehlers $SL(2,\mathbb{R})$ group is a symmetry group acting on a certain solutions possessing one Killing vector. When combined with the Matzner-Misner group, it leads to an infinite dimensional symmetry know as the Geroch group acting on solutions of Einstein equations with two commuting Killing vectors (axisymmetric stationary solutions) [6]. These results aslo provided a direct link with the integrability of these theories in the reduction of two dimensions, i.e. the existence of Lax pairs for the corresponding equations of motion [7].

Motivated by the dimensional reduction, it has been argued that the Kac-Moody algebra $\bar{\mathfrak{g}}^{++}$ defined in Section 2.4.1 (overextension of $\bar{\mathfrak{g}}$) can play a role in the compactification to 1 dimension [23]. Finally, when all the dimensions are compactified, it is obvious to extend the algebra $\bar{\mathfrak{g}}^{++}$ to $\bar{\mathfrak{g}}^{+++}$, defined in Section 2.4.1, (triple extension of $\bar{\mathfrak{g}}$) by adding a third vertex. Such $\mathcal{G}^{+++}$ symmetries were first conjectured in the aforementioned cases [21, 24] and the extension to all $\mathcal{G}^{+++}$ was proposed in [25]. So this construction motivates the fact that the pure gravity in D dimensions could have the symmetry A_{D-3}^{+++} and that the eleven-dimensional supergravity could have the symmetry $\bar{\mathfrak{g}}^{+++} = E_8^{+++} = E_{11}$ (see Fig. 3.4).

Similarly other well known theories would also have symmetries under Kac-Moody algebras. Indeed the effective action of the bosonic string in 26 dimensions would possess a very-extended symmetry D_{24}^{+++}. More generally all simple maximally non-compact Lie groups $\mathcal{G}$ could be generated from the reduction down to 3 dimensions of suitably chosen actions [26] and it was conjectured that these actions possess the very-extended Kac-Moody symmetries $\mathcal{G}^{+++}$ [25]. $\bar{\mathfrak{g}}^{+++}$ algebras are defined by the Dynkin diagrams depicted in Fig. 3.4, obtained from those of $\bar{\mathfrak{g}}$ by adding three nodes.

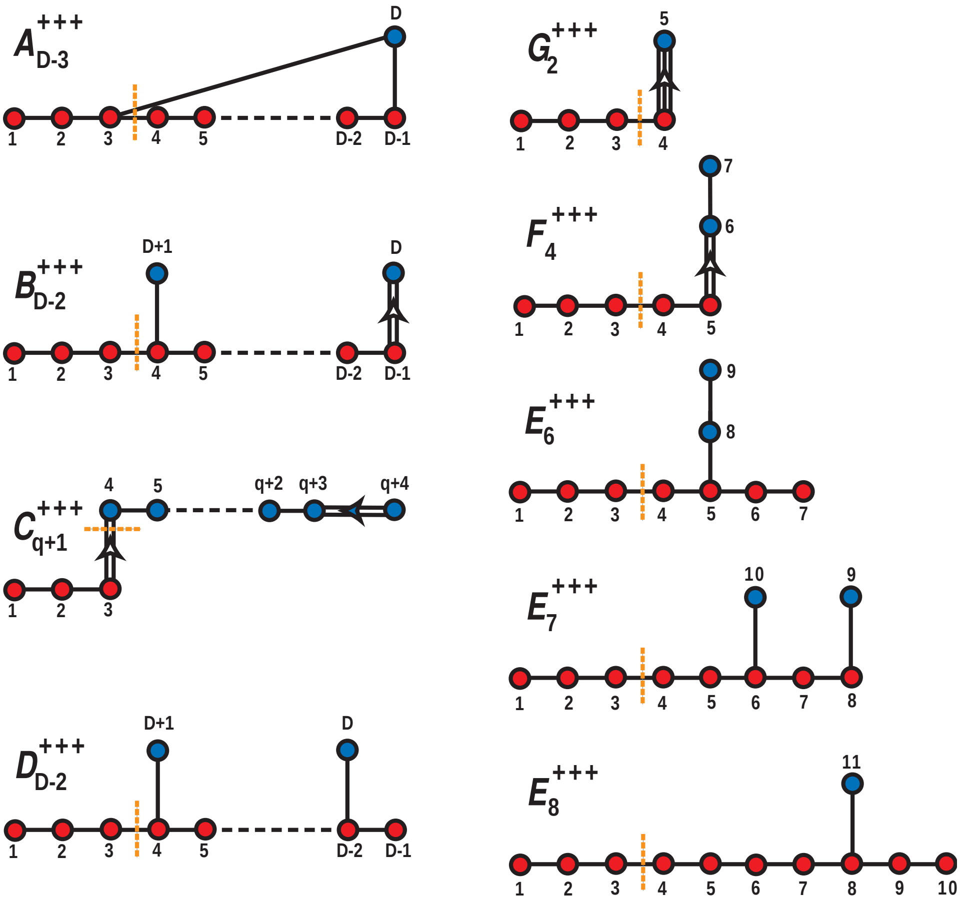

Figure 3.4: The nodes labelled 1, 2, 3 define the Kac-Moody extensions $\bar{\mathfrak{g}}^{+++}$ of the Lie algebras $\bar{\mathfrak{g}}$. The horizontal line starting at 1 defines the 'gravity line', which is the Dynkin diagram of a A_{D-1} subalgebra.

Chapter 4

σ-models for Lorentzian Kac-Moody Algebras

In the precedent section, we have seen how hidden Kac-Moody symmetries are exhibited through compactifications. Now, we would like to make these Kac-Moody symmetries manifest and in this context, it is interesting to construct an action, $\mathcal{S}_{\mathcal{G}^{+++}}$, explicitly invariant under the infinite-dimensional group $\mathcal{G}^{+++}$ [27]. This action is defined in a reparametrisation invariant way on a world-line, a priori unrelated to space-time, in terms of an infinity of fields $\psi_a(\xi)$ where ξ spans the world-line. The fields $\psi_a(\xi)$ live in a coset space $\mathcal{G}^{+++}/\mathcal{K}^{+++}$ where the subgroup $\mathcal{K}^{+++}$ is invariant under a *temporal involution* Ω_1 (defined in Section 4.2) which ensures that the action is $SO(1, D-1)$ invariant and which allows identification of index 1 to a time coordinate.

4.1 Level Decomposition

In the same way that in the case of the construction of the coset Lagrangian $\mathcal{L}_{\mathcal{G}/\mathcal{K}}$ in Section 3.6, we can write an element of the coset by exponentiating the Borel subalgebra $\mathfrak{b}$ of $\bar{\mathfrak{g}}^{+++}$ as:

$$\mathcal{V}(\xi) = exp^{\mathcal{B}^a \psi_a(\xi)} \,. \tag{4.1.1}$$

To each Borel generator $\mathcal{B}^a$, we associate a field $\psi_a(\xi)$. As there is an infinity of generators $\mathcal{B}^a$, there is an infinite number of fields $\psi_a(\xi)$. So we have to organise this summation on the infinity of fields in such a way that a recursive approach has a sense. In this context, we introduce a *level decomposition* with respect to the finite subalgebra A_{D-1}. Each $\bar{\mathfrak{g}}^{+++}$ contains indeed a subalgebra $\mathfrak{gl}(D)$ such that $\mathfrak{sl}(D) \subset \mathfrak{gl}(D) \subset \bar{\mathfrak{g}}^{+++}$.

In the particular case of E_8^{+++}, the level l counts the number of times the simple root α_{11} not contained in the gravity line (see Figure 3.4) appears in irreducible representation of A_{10} [28]: $\alpha = l\alpha_{11} + \sum_{i=1}^{10} a_i \alpha_i$ (where the α_i are simple roots of the gravity line). Using this mechanism, we find a nice decomposition of the infinite number of generators. At each level we have a finite number of generators and fields associated to it. The symmetry properties of these generators are fixed by Young tableaux describing an irreducible representation appearing at a given level.

Exploring the decomposition of E_8^{+++} into representations of A_{10}, the positive root generators at level 0, 1, 2 and 3 are respectively, $K^a{}_b$ ($\mathfrak{gl}(11)$ generators satisfying commutation relations: $[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b$), $R^{a_1 a_2 a_3}$, $R^{a_1 a_2 a_3 a_4 a_5 a_6}$ and $R^{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8, b}$. The level 1 and 2 tensors are antisymmetric and the level 3 tensor is associated to a mixed Young tableau (with the constraint $R^{[a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8, b]} = 0$). The fields associated to these generators are respectively $h_a^b(\xi)$, $A_{a_1 a_2 a_3}(\xi)$, $A_{a_1 a_2 a_3 a_4 a_5 a_6}(\xi)$ and $A_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8, b}(\xi)$.

At higher levels (> 3), there is an infinite number of representations characterised by some Young tableaux.

With this decomposition, it is possible to rewrite the coset representative $\mathcal{V}(\xi)$ in (4.1.1) as

$$\mathcal{V}(\xi) = exp\Big(\underbrace{\sum_{a\geq b} h_b{}^a(\xi) K^b{}_a}_{Level\ 0}\Big) exp\Big(\underbrace{\sum A_{a_1\,a_2\,a_3}(\xi)\, R^{a_1\,a_2\,a_3} + \dots}_{Level\ \geq 1}\Big)\,, \tag{4.1.2}$$

where the first exponential contains only level zero operators and the second one the positive root generators of levels strictly greater than zero.

4.2 The Temporal Involution and the Coset $\mathcal{G}^{+++}/\mathcal{K}^{+++}$

The metric $g_{\mu\nu}$ at a fixed space-time point parametrises the coset $GL(D)/SO(1, D-1)$. To construct a $\mathcal{G}^{+++}$-invariant action containing such a tensor, we shall build a non linear realisation of $\mathcal{G}^{+++}$ in a coset space $\mathcal{G}^{+++}/\mathcal{K}^{+++}$ where the subgroup $\mathcal{K}^{+++}$ contains the Lorentz group $SO(1, D-1)$. As $SO(1, D-1)$ is non-compact, we cannot use the Chevalley involution ω to construct $\mathcal{K}^{+++}$ that is now non-compact[1]. Rather we will use the *temporal involution* Ω_1 from which the required non-compact generators of $\mathcal{K}^{+++}$ can be selected.

The temporal involution Ω_1 acts on the Chevalley generators $\{h_i, e_i, f_i\}$ in the following way:

$$\begin{aligned} & \quad\ \Omega_1 & \\ h_i &\ \rightarrow\ -h_i\,, & (4.2.1)\\ e_i &\ \rightarrow\ -\epsilon_i\, f_i\,, & (4.2.2)\\ f_i &\ \rightarrow\ -\epsilon_i\, e_i\,. & (4.2.3)\end{aligned}$$

Here e_i is expressed as A_{D-1} tensors and f_i as the tensor with upper and lower indices interchanged. ϵ_1 is defined as $+1(-1)$ if the number of $'1'$ indices (that is the number of time indices) is even (odd). In the same way, all the generators e_α and f_α (obtained by multiple commutators of e_i and f_i as in (2.2.4)) are mapped under the temporal involution to $(e_\alpha, f_\alpha) \longrightarrow -\epsilon_\alpha\,(f_\alpha, e_\alpha)$ where ϵ_α is equal to $+1(-1)$ depending that the number of indices $'1'$ are even (odd). As all the generators of $\bar{\mathfrak{g}}^{+++}$ are expressed in terms of $K^a{}_b, R_u$ and tensors $R_{d_1\dots d_s}{}^{c_1\dots c_r}$ (see Section 4.1), we can write the effect of the temporal involution on these quantities as

$$\begin{aligned} & \quad\ \Omega_1 & \\ K^a{}_b &\ \rightarrow\ -\epsilon_a\,\epsilon_b\, K^b{}_a\,, & (4.2.4)\\ R_u &\ \rightarrow\ -R_u\,, & (4.2.5)\\ R_{d_1\dots d_s}{}^{c_1\dots c_r} &\ \rightarrow\ -\epsilon_{c_1}\dots\epsilon_{c_r}\,\epsilon_{d_1}\dots\,\epsilon_{d_s}\,\bar{R}_{c_1\dots c_r}{}^{d_1\dots d_s} & (4.2.6)\end{aligned}$$

with $\epsilon_a = -1$ if $a = 1$ and $\epsilon_a = 1$ otherwise and $\bar{R}_{c_1\dots c_r}{}^{d_1\dots d_s}$ is the negative step operator obtained by interchanging the lower and upper indices of the positive one. For example: $\Omega_1(R^{9\,10\,11}) = -\epsilon_9\,\epsilon_{10}\,\epsilon_{11}\,R_{9\,10\,11} = -R_{9\,10\,11}$.

The subgroup $\mathcal{K}^{+++}$ of $\mathcal{G}^{+++}$ is defined as the subgroup invariant under this involution. His generators are $(e_\alpha - \epsilon_\alpha f_\alpha)$.

[1]The Chevalley involution ω: $(h_i, e_i, f_i) \longrightarrow -(h_i, e_i, f_i)$ left the sign of all compact generators invariant.

4.3 Construction of the Action $\mathcal{S}_{\mathcal{G}^{+++}}$

Defining

$$\frac{dv(\xi)}{d\xi} = \frac{d\mathcal{V}}{d\xi}\mathcal{V}^{-1} \ , \qquad \frac{d\tilde{v}(\xi)}{d\xi} = -\Omega_1 \frac{dv(\xi)}{d\xi} = \tilde{\mathcal{V}}^{-1}\frac{d\tilde{\mathcal{V}}}{d\xi} \ , \qquad \mathcal{P} = \frac{1}{2}\left(\frac{dv}{d\xi} + \frac{d\tilde{v}}{d\xi}\right), \quad (4.3.1)$$

where Ω_1 is the temporal involution which allows identification of index 1 to a time coordinate, one obtains in terms of the ξ-dependent fields, an action $\mathcal{S}_{\mathcal{G}^{+++}}$ invariant under global $\mathcal{G}^{+++}$ transformations, defined on the coset $\mathcal{G}^{+++}/\mathcal{K}^{+++}$:

$$\mathcal{S}_{\mathcal{G}^{+++}} = \int d\xi \, \frac{1}{n(\xi)} \big(\mathcal{P} \mid \mathcal{P}\big) \, , \qquad (4.3.2)$$

where $n(\xi)$ is an arbitrary lapse function ensuring reparametrisation invariance on the world-line and $(\cdot|\cdot)$ is the $\mathcal{G}^{+++}$ invariant bilinear form ensuring invariance of the action (4.3.2) under the global transformations $\mathcal{G}^{+++}$.

The fundamental question is now : is there a link between the action invariant under the Kac-Moody algebras and the action of the space-time theory? In particular, we would like to relate the action invariant under $\mathcal{E}_8^{+++}$ and eleven-dimensional supergravity. To do that, we must interpret the parameter ξ which the fields of the σ-model depend on and find the significance of the infinity of fields $\psi_a(\xi)$. In this context, we are going to study the Kac-Moody algebra $\bar{\mathfrak{g}}^{++}$.

4.4 From a $\mathcal{G}^{+++}$- to a $\mathcal{G}^{++}$-Invariant Action

To make connections between this new formalism and the covariant space-time theories, it is interesting to analyse the several actions invariant under overextended Kac-Moody algebra $\bar{\mathfrak{g}}^{++}$ (double extension of $\bar{\mathfrak{g}}$). The Dynkin diagram of $\bar{\mathfrak{g}}^{++}$ is obtained by deleting the root α_1 in the Dynkin diagram of actions invariant under overextended Kac-Moody algebra $\bar{\mathfrak{g}}^{+++}$. The $\mathcal{G}^{+++}$-invariant action leads to two distinct theories: $\mathcal{G}_C^{++}$ and $\mathcal{G}_B^{++}$ obtained both by a truncation of a infinity of $\mathcal{G}^{+++}$-fields, putting to zero all the fields multiplying generators involving the deleted root α_1 in Figure 3.4. This truncation is realised consistently with all equations of motion, i.e. it implies that all the solutions of the equations of motions of $\mathcal{S}_{\mathcal{G}^{++}}$ are also solutions of the equations of motion of $\mathcal{S}_{\mathcal{G}^{+++}}$ [29].

4.4.1 $\mathcal{G}_C^{++}$-Invariant Action

The recent study of the properties of cosmological solutions in the vicinity of a space-like singularity revealed an overextended symmetry $\mathcal{G}_C^{++}$. The action $\mathcal{S}_{\mathcal{G}_C^{++}}$ restricted to a defined number of lowest levels is equal to the corresponding space-time theory in which the fields depend only on the time coordinate (see Table 4.1) [12, 13, 30]. The parameter ξ is defined as a time coordinate and this $\mathcal{G}_C^{++}$-theory carries a Euclidean signature in $D-1$ dimensions.

	Fields belonging to $\mathcal{S}_{\mathcal{E}_8^{++}}$		*Fields of supergravity depending on time*
level 0	$g_{\hat{\mu}\hat{\nu}}(t) = f(h_a{}^b)$	$\leadsto$	metric
level 1	$A_{\hat{\mu}\hat{\nu}\hat{\rho}}(t)$	$\leadsto$	3-form electric potential
level 2	$A_{\hat{\mu}_1\ldots\hat{\mu}_6}(t)$	$\leadsto$	6-form magnetic potential (dual of the 3-form)
level 3	$A_{\hat{\mu}_1\ldots\hat{\mu}_8,\hat{\nu}}(t)$	$\leadsto$	'dual' of the metric

Table 4.1: Link between the lowest level up to level 3 of the cosmological $\mathcal{E}_8^{++}$-invariant theory and the fields of the eleven-dimensional supergravity.

4.4.2 $\mathcal{G}_B^{++}$-Invariant Action

The action $\mathcal{S}_{\mathcal{G}^{+++}}$ contains another $\mathcal{G}^{++}$-invariant action $\mathcal{S}_{\mathcal{G}_B^{++}}$, obtained by performing the same consistent truncation as the one for $\mathcal{G}_C^{++}$, but now performed after a $\mathcal{G}^{+++}$ Weyl transformation. As the Weyl transformations modify the identification of the indices 1 and 2 which become respectively space and time (see Section 4.5), the resulting action $\mathcal{S}_{\mathcal{G}_B^{++}}$ is different from $\mathcal{S}_{\mathcal{G}_C^{++}}$. This gives an action $\mathcal{S}_{\mathcal{G}_B^{++}}$ with a Lorentz signature for the metric and with the parameter ξ identified to the missing space coordinate instead of t. This $\mathcal{G}_B^{++}$-theory admits exact solutions identical to the ones of covariant Einstein and field equations describing intersecting extremal brane solutions smeared in all direction but one (see Table 4.2) [27, 31].

	Fields belonging to $\mathcal{S}_{\mathcal{E}_8^{++}}$		*Branes of M-theory*
level 0	$g_{\hat{\mu}\hat{\nu}}(x)$	$\rightsquigarrow$	KK-wave (0-brane)
level 1	$A_{\hat{\mu}\hat{\nu}\hat{\rho}}(x) + g_{\hat{\mu}\hat{\nu}}(x)$	$\rightsquigarrow$	$M2$ (2-brane)
level 2	$A_{\hat{\mu}_1\ldots\hat{\mu}_6}(x) + g_{\hat{\mu}\hat{\nu}}(x)$	$\rightsquigarrow$	$M5$ (5-brane)
level 3	$A_{\hat{\mu}_1\ldots\hat{\mu}_8,\hat{\nu}}(x) + g_{\hat{\mu}\hat{\nu}}(x)$	$\rightsquigarrow$	$KK6$-monopole

Table 4.2: Link between the lowest level up to level 3 of the brane $\mathcal{E}_8^{++}$-invariant theory and the branes of M-theory.

Moreover, intersections rules are neatly encoded in $\mathcal{G}_B^{++}$ by orthogonality conditions on the positive real roots characterising each branes.

4.5 Weyl Transformations and Their Consequences

In this section, we review some consequences of Weyl transformations. First, a Weyl transformation on a generator T of $\bar{\mathfrak{g}}^{+++}$ can be expressed as a conjugaison by a group element U_W of $\bar{\mathfrak{g}}^{+++}$: $T \longrightarrow U_W T U_W^{-1}$. Because of the non-commutativity of Weyl transformation with the temporal involution Ω:

$$U_W \Omega T U_W^{-1} = \Omega' U_W T U_W^{-1} \,, \tag{4.5.1}$$

different Lorentz signatures (t, s) (where $t\,(s)$ is the number of time (space) coordinates) can be obtained [29]. More precisely, Weyl reflections of the gravity line do not change the global Lorentz signature (t, s) but it change only the identification of time coordinates (see the example developed below). In fact, only Weyl reflections with respect to roots not belonging to the gravity line can change the global signature of the theory.

For instance the signatures found for E_8^{+++} are $(1, 10)$, $(2, 9)$, $(5, 6)$, $(6, 5)$ and $(9, 2)$ [32]. These signatures match perfectly with the signatures changing dualities. Indeed, we can interpret the Weyl transformation with respect to root α_{11} (not belonging to gravity line) as a double T-duality in the direction 9 and 10 with an exchange of these directions [25, 33]. Moreover, these signatures match also with the exotic phases of M-theories (M' and M*) [34].

The previous construction has been generalized and the signatures for all $\bar{\mathfrak{g}}^{+++}$ have been found [35, 36].

Example: Effect of the Weyl reflection s_1

To illustrate the consequence of the non-commutativity of Weyl transformations with the temporal involution Ω_1 (4.5.1), let us consider the simple example of Weyl reflection associated to the root α_1: s_1 defined in (2.2.90). Only simple roots α_1 and α_2 are modified by

this reflection:

$$
\begin{aligned}
s_1(\alpha_1) &= -\alpha_1\,, &(4.5.2)\\
s_2(\alpha_2) &= \alpha_2 + \alpha_1\,, &(4.5.3)
\end{aligned}
$$

and this reflection leaves the other simple roots unchanged ($s_1(\alpha_i) = \alpha_i$ for i$\neq$ 1, 2). Let us consider now, the action of this Weyl reflection on the generators associated to these roots. The positive generators associated to the simple roots of the gravity line are expressed as:

$$
e_i = K^i{}_{i+1} \qquad i = 1, \ldots, D-1\,. \tag{4.5.4}
$$

The generators associated to the roots α_1 and α_2 are modified respectively as[2]

$$
\begin{aligned}
U\,K^1{}_2\,U^{-1} &= \delta\,K^2{}_1\,, &(4.5.5)\\
U\,K^2{}_3\,U^{-1} &= \rho\,[K^1{}_2, K^2{}_3] = \rho\,K^1{}_3\,, &(4.5.6)
\end{aligned}
$$

where δ and ρ are plus or minus signs which arise as positive generators are representations of the Weyl group up to signs. We will see that such signs always cancel in the determination of Ω' because they are identical in the Weyl transform of corresponding positive and negative roots, as their commutator is in the Cartan subalgebra which forms a true representation of the Weyl group.

Before the Weyl reflection, the index 1 is time index and the others are space indices. We recall that the action on the involution Ω on the generators $K^a{}_b$ (see (4.2.4)) is

$$
\Omega\,K^a{}_b = -\,\epsilon_a\,\epsilon_b\,K^b{}_a\,, \tag{4.5.7}
$$

where ϵ_a is equal to $1(-1)$ if the index a is space (time). If we apply (4.5.1), we find the action on Ω' on the two generators:

$$
\begin{aligned}
\delta\,\Omega'\,K^1{}_2 &= \Omega'\,U\,K^2{}_1\,U^{-1} = U\,\underbrace{\Omega_1\,K^2{}_1}_{K^1{}_2}\,U^{-1}\,, &(4.5.8)\\
&= \delta\,K^2{}_1\,, &(4.5.9)\\
\rho\,\Omega'\,K^2{}_3 &= \Omega'\,U\,K^1{}_3\,U^{-1} = U\,\underbrace{\Omega_1\,K^1{}_3}_{K^3{}_1}\,U^{-1}\,, &(4.5.10)\\
&= \rho\,K^3{}_2\,. &(4.5.11)
\end{aligned}
$$

One gets

$$
\begin{aligned}
\Omega'\,K^1{}_2 = K^2{}_1 \quad &\mathit{and} \quad \Omega_1\,K^1{}_2 = K^2{}_1\,, &(4.5.12)\\
\Omega'\,K^2{}_3 = K^3{}_2 \quad &\mathit{and} \quad \Omega_1\,K^2{}_3 = -\,K^3{}_2\,. &(4.5.13)
\end{aligned}
$$

The other positive generators associated to simple roots are invariant under this Weyl transformation:

$$
\Omega'\,K^n{}_{n+1} = \Omega_1\,K^n{}_{n+1} = -\,K^{n+1}{}_n \qquad n = 3, \ldots, D-1\,. \tag{4.5.14}
$$

The content of (4.5.12), (4.5.13) and (4.5.14) is represented in Table 4.3. The signs below the generators of the gravity line indicate the sign in front of the negative step operator $K^{a+1}{}_a$ obtained by the involution Ω_1 (Ω') : a minus sign is in agreement with the Chevalley involution and indicates that the indices in $K^a{}_{a+1}$ are both either space or time indices while a plus sign indicates that one index must be time and the other space.

[2] A Weyl transformation on a generator T of $\bar{\mathfrak{g}}^{+++}$ can be expressed as a conjugaison by a group element U_W of $\bar{\mathfrak{g}}^{+++}$: $T \longrightarrow U_W T U_W^{-1}$.

	$K^1{}_2$	$K^2{}_3$	$\dots$	$K^{D-1}{}_D$	Time index
Ω_1	+	-	-	-	1
Ω'	+	+	-	-	2

Table 4.3: Involution switch from Ω_1 to Ω' due to the Weyl reflection s_1.

Table 4.3 shows that the time coordinate after the Weyl reflection (see the line Ω') must be identified either with 2, or with all indices $\neq 2$. We choose the first description, which leaves unaffected coordinates attached to the planes invariant under the Weyl transformation[3].

[3] The Weyl reflection ω_α fixes the hyperplane orthogonal to α (see (2.2.95).)

Chapter 5

Conclusions and Perspectives

We have seen that a large amount of physical information is neatly encoded in Kac-Moody algebras (cosmological solutions, branes, intersection rules, T-duality, etc.). So the study of $\mathcal{G}^{++}$ and $\mathcal{G}^{+++}$ constitutes an interesting approach to understand M-theory which is conceptually different from the Einstein approach. We have now to answer the fundamental question of this approach: are these symmetries only a consequence of the compactification process or are they also symmetries of the uncompactified theory? This question is related to the role played by the infinite number of fields. We have seen in previous chapters that only the fields of level < 3 have a clear physical interpretation.

Indeed, we have seen in Section 4.4.1, that the E_{10} σ-model limited to the roots up to level 3 reveals a perfect match with the bosonic equations of motion of eleven-dimensional supergravity in the vicinity of a spacelike singularity, where fields depend only on time. It was conjectured that spatial derivatives are hidden in some higher level fields of the σ-model [11].

In Section 4.4.2, we have studied an alternative E_{10} σ-model. It yields all the basic BPS solutions of eleven-dimensional supergravity namely the KK-wave, the $M2$-brane, the $M5$-brane and the $KK6$-monopole smeared in all space dimensions but one as well as their exotic counterparts.

Elucidating the role of the infinite number of fields of E_{10} and E_{11} generators is an important problem in the Kac-Moody approach of M-theory. Various propositions have been developed to give an interpretation of fields of increasing level. Recently, an infinite number of BPS states for eleven-dimensional supergravity were constructed [37]. For each positive real root of $E_9 \subset E_{10}$, a BPS solution of eleven-dimensional supergravity, or of its exotic counterparts, was obtained, depending on two non-compact transverse spatial variables. All such solutions are related by U-dualities, realized via E_9 Weyl transformations. Moreover, for real roots of E_{10}, which are not roots of E_9, additional BPS solutions, transcending 11-dimensional supergravity, were found. For example, an explicit BPS solution attached to a level 4 root of E_{10} was obtained in agreement with reference [38], and describes the $M9$-brane solution in eleven dimensions which corresponds to the uplifting of the $D8$-brane of massive type IIA supergravity [39, 40]. This result suggests that E_{10} also describes a massive version of eleven-dimensional supergravity.

Some very promising results have also been obtained regarding the incorporation of quantum corrections into the structure of the Kac-Moody algebras related to string and M-theory [41–43].

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