Lectures on Two-dimensional Conformal Field Theory and Perturbative String Theory

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Abstract

These lectures are meant to be an introduction to 2D CFT tools and methods used in string theory.
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Overview

Two-dimensional Conformal Field Theory is a very vast topic. I had to make a very drastic selection of the topics I will discuss. Since these lectures are intended for string theorists that might be more familiar with “spacetime” language rather than with “worldsheet” language, I chose to give an overview of the 2D CFT tools that are relevant to describe the objects string theorists are familiar with. The content of the four lectures will be the following:

1. Introduction. Generic results. The free boson.
2. Famous CFTs.
3. CFT on non-trivial worldsheets.
4. Supersymmetric CFTs.

In a spacetime-language, this would roughly translate to:

1. Perturbative string theory in flat spacetime.
2. Perturbative string theory in curved spacetime.
3. Loop amplitudes, D-branes.
4. Superstrings.

For a very efficient introduction to stringy-oriented 2D CFT, one can read the relevant chapter in Kiritsis’ book “String Theory in a Nutshell”. If you want to learn more, the bible of 2D CFT is the yellow book of Di Francesco, Mathieu and Senechal “Conformal Field Theory”.
Chapter 1

Introduction to Two-dimensional Conformal Field Theory

1.1 Quantum Strings and 2D CFT

Consider a string moving in a spacetime with metric $G_{\mu\nu}$. The action naturally associated to this string is proportional to the area of the surface spanned by the string. This is the Nambu-Goto action:

$$ S_{NG} = -T \int d^2 \xi \sqrt{-\det (G_{\mu\nu}(X^\rho)\partial_\mu X^\nu X^\rho)} \quad (1.1.1) $$

where $T$ is the string tension, related to the square of the string length $\alpha'$:

$$ T = \frac{1}{2\pi \alpha'} \quad (1.1.2) $$

In order to quantize this action, it is convenient to introduce a worldsheet metric $g^{ij}$. We can rewrite the NG action as:

$$ S_P = -\frac{1}{4\pi \alpha'} \int d^2 \xi \sqrt{-\det g^{ij} G_{\mu\nu}(X^\rho)\partial_\mu X^\nu X^\rho} \quad (1.1.3) $$

This action is called the Polyakov action in the stringy literature. It is also known as the non-linear sigma-model on the target space with metric $G_{\mu\nu}$. This action has a local invariance under worldsheet diffeomorphism. There is also another symmetry called Weyl invariance:

$$ g^{ij} \rightarrow \Lambda(\xi) g^{ij} \quad (1.1.4) $$

In order for the quantum theory to make sense, it is important for these local symmetries to be preserved at the quantum level. A quantum field theory invariant under the Weyl rescaling (1.1.4) is called a conformal field theory. Conformal invariance on the worldsheet implies the vanishing of the beta-functions. The beta-function associated with the coupling $G_{\mu\nu}$ in the Polyakov action can be computed perturbatively in $\alpha'$:

$$ \beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + O(\alpha'^2) \quad (1.1.5) $$

Conformal invariance on the worldsheet implies Einstein equations in spacetime (!). The previous argument tells us that strings may only be quantized in spacetime satisfying (generalized) Einstein equations. These spacetimes are the vacua of string theory.
In each one of these vacua, the fluctuations of the strings are coded by a 2D CFT. More precisely the vector space of states of a quantum string is a (subset of the) Hilbert space of a 2D CFT. Scattering amplitudes of strings can be computed as correlation functions in this CFT.

1.2 Basic facts about 2D CFTs

After this stringy introduction we will turn to the study of 2D CFTs. In this lecture we will focus on the basic consequences of conformal invariance.

1.2.1 Conformal invariance in two dimensions

By definition a conformal transformation leaves the metric tensor invariant up to a scale:

\[ x \rightarrow x', \quad g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x) \] (1.2.1)

Such transformations have the property of preserving angles. In a generic number of dimensions the conformal group has finite dimension (it is isomorphic to \( SO(d + 1, 1) \) in an euclidean spacetime of dimension \( d \)). However in two dimensions the conformal group has infinite dimension. Let us consider the two-dimensional plane parameterized by complex coordinates \( z, \bar{z} \). The metric reads:

\[ ds^2 = dzd\bar{z} \] (1.2.2)

Then any holomorphic redefinition of \( z \) and any anti-holomorphic redefinition of \( \bar{z} \) are conformal transformations:

\[ z \rightarrow f(z), \quad \bar{z} \rightarrow \bar{f}(\bar{z}) \]

\[ ds^2 = dzd\bar{z} \rightarrow ds'^2 = f'(z)\bar{f}'(\bar{z})dzd\bar{z} \] (1.2.3)

One may feel uncomfortable to perform independent redefinitions of \( z \) and \( \bar{z} \). The good way to think about it is the following: we think of \( z \) and \( \bar{z} \) as independent complex variables, so our theory is defined on a two-dimensional complex space. The physical slice in this space is the subspace on which the complex conjugate of \( z \) is equal to \( \bar{z} \).

Conformal invariance in two dimensions puts a lot of constraints on the structure of a quantum field theory. In the following we will try to use this huge symmetry to simplify the study of 2D CFTs.

1.2.2 OPEs

Generically in a quantum field theory one can expand the product of two operators as follows:

\[ \Phi_1(z, \bar{z})\Phi_2(0, 0) = \sum_i C_{12i}z^{h_{12}z}z^{h_{12}i}\Phi_i(0, 0) \] (1.2.4)

where the sum is performed over a basis of the operators of the theory. The knowledge of these expansions is extremely powerful since it allows for the iterative computation of correlation functions, reducing \( n \)-point functions to \( n - 1 \) point functions. In 2D CFTs generically one can get some control over this operator algebra, and use it to solve the theory. In the following one of our main concerns will be to compute OPE’s.
1.2.3 Primary fields

Let us consider a two-dimensional quantum field theory invariant under conformal transformations. A field is called a primary field if it transforms in a “natural” way under conformal transformations:

\[ z \rightarrow f(z) \]
\[ \bar{z} \rightarrow \bar{f}(\bar{z}) \]
\[ \phi(z, \bar{z}) \rightarrow \tilde{\phi}(f(z), \bar{f}(\bar{z})) = f'(z)^{-\Delta} \bar{f}'(\bar{z})^{-\bar{\Delta}} \phi(z, \bar{z}) \quad (1.2.5) \]

where \( \Delta \) and \( \bar{\Delta} \) are called holomorphic and anti-holomorphic conformal dimensions of the field \( \phi \). As we will see primary fields play a central role in the study of 2D CFTs (and an even more important role in string theory).

1.2.4 Conformal Ward identity

As a first example of a purely \( \mathcal{Q} \) computation, let us consider the Ward identity implied by conformal invariance for correlation functions of primary fields. The infinitesimal form of the transformation law \([1.2.5]\) reads:

\[ z \rightarrow z + \alpha(z) \]
\[ \bar{z} \rightarrow \bar{z} + \bar{\alpha}(\bar{z}) \]
\[ \phi(z, \bar{z}) \rightarrow \tilde{\phi}(z + \alpha(z), \bar{z} + \bar{\alpha}(\bar{z})) = (1 - \Delta \alpha'(z))(1 - \bar{\Delta} \bar{\alpha}'(\bar{z})) \phi(z, \bar{z}) \quad (1.2.6) \]

where \( \alpha(z) \) and \( \bar{\alpha}(\bar{z}) \) are infinitesimal.

We consider the correlation function of \( n \) primary fields \( \phi_1(z_1, \bar{z}_1), \ldots, \phi_n(z_n, \bar{z}_n) \):

\[ \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle = \frac{\int \mathcal{D}X e^{-S[X]} \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n)}{\int \mathcal{D}X e^{-S[X]}} \quad (1.2.7) \]

By definition of the stress-energy tensor \( T^{\mu\nu} \), the variation of the action under a reparametrization \( x^\mu \rightarrow x'^\mu + \epsilon^\mu(x) \) reads:

\[ \delta S = -\frac{1}{2\pi} \int dx^\mu \epsilon^\nu(x) T_{\mu\nu} \quad (1.2.8) \]

where the contour integral is performed around the support of the function \( \epsilon^\mu \).

Thus invariance of the correlation function under conformal transformation implies:

\[ \frac{1}{2\pi} \int dz (\alpha(z) \langle T_{zz}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle + \bar{\alpha}(\bar{z}) \langle T_{\bar{z}\bar{z}}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle) \]
\[ - \frac{1}{2\pi} \int d\bar{z} (\alpha(\bar{z}) \langle T_{\bar{z}\bar{z}}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle + \bar{\alpha}(\bar{z}) \langle T_{\bar{z}\bar{z}}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle) \]
\[ = \sum_{i=1}^n (\Delta_i \alpha'(z_i) + \alpha(z_i) \partial_{z_i} + \bar{\Delta}_i \bar{\alpha}'(\bar{z}_i) + \bar{\alpha}(\bar{z}_i) \partial_{\bar{z}_i}) \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle \]
\[ (1.2.9) \]

The previous equation is quite generic. First we will extract some information concerning the stress-energy tensor. Notice that the previous equality does not depend on the choice of integration contour. Thus by subtracting the same equation for two slightly different contours, we can get rid of the right-hand side. Using Stokes’ theorem we obtain:

\[ \int d^2z (\partial_2 (\alpha(z) \langle T_{zz}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle + \bar{\alpha}(\bar{z}) \langle T_{\bar{z}\bar{z}}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle)
\]
\[ + \partial_2 (\alpha(\bar{z}) \langle T_{\bar{z}\bar{z}}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle + \bar{\alpha}(\bar{z}) \langle T_{\bar{z}\bar{z}}(z, \bar{z}) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle)] = 0 \]
\[ (1.2.10) \]
Then we take special values for the functions $\alpha, \bar{\alpha}$. Choosing $\alpha(z) = a + b z$ and $\bar{\alpha}(\bar{z}) = \bar{a} + \bar{b} \bar{z}$, we obtain the following equations:

$$\partial_z T_{zz} + \partial_{\bar{z}} T_{z\bar{z}} = 0$$
$$\partial_{\bar{z}} T_{z\bar{z}} + \partial_z T_{zz} = 0$$
$$z(\partial_z T_{zz} + \partial_{\bar{z}} T_{z\bar{z}}) + T_{zz} = 0$$
$$\bar{z}(\partial_{\bar{z}} T_{z\bar{z}} + \partial_z T_{zz}) + T_{z\bar{z}} = 0$$

from which we deduce the tracelessness of the stress-energy tensor:

$$\text{Tr}(T) = T_{zz} = 0$$

as well as the holomorphy (resp anti-holomorphy) of the $zz$ component of the stress-tensor.

We obtain the local Ward identity for correlation functions of primary fields:

$$\langle T_{zz}(z, \bar{z})\phi(z_1, \bar{z}_1)\ldots\phi_n(z_n, \bar{z}_n) \rangle = \sum_{i=1}^n \left( \frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i} \right) \alpha(z) \langle \phi(z_1, \bar{z}_1)\ldots\phi_n(z_n, \bar{z}_n) \rangle$$

We deduce the OPE between the holomorphic stress-tensor $T$ and a primary field $\phi$ of conformal dimension $\Delta$:

$$T(z)\phi(w, \bar{w}) = \frac{\Delta}{(z-w)^2} \phi(w, \bar{w}) + \frac{1}{z-w} \partial \phi(w, \bar{w}) + \ldots$$

The factorization between the holomorphic and anti-holomorphic behavior that we noticed during this computation is actually generic in 2D CFTs. Each operator can be split into a holomorphic and an anti-holomorphic factor. In the following we will mainly focus on the holomorphic side of the theory. The gluing between the holomorphic and anti-holomorphic sectors will be briefly discussed later (lecture 3).

### 1.2.5 The free boson

Before going further into the structure of 2D CFTs, let us investigate a first example: the free boson. The action is:

$$S = -\frac{1}{4\pi} \int d^2 z \partial X \bar{\partial} X$$

Comparing with the Polyakov action (1.1.3), we see that this theory will be relevant for the quantization of strings in flat spacetime. The equations of motion reads:

$$\bar{\partial} \partial X = 0$$
We read from the action that the field $\partial X$ has dimension one, so it should not be surprising that the two-point function reads:

$$\langle \partial X(z, \bar{z})\partial X(0, 0) \rangle = -\frac{2}{z^2} \quad (1.2.19)$$

from which we deduce the OPE (a possible first order pole vanishes by symmetry):

$$\partial X(z, \bar{z})\partial X(0, 0) = -\frac{2}{z^2} + \mathcal{O}(1) \quad (1.2.20)$$

The holomorphic stress-tensor is:

$$T(z) = -\frac{1}{4} : \partial X \partial X : \quad (1.2.21)$$

The field $\partial X$ is a primary field of conformal dimension one. To prove this we compute the OPE of this field with the stress-tensor, using Wick’s theorem:

$$\partial X(0)T(z) = -\frac{1}{4} : \partial X(z) \partial X(z) :$$

$$= \frac{1}{z^2} \partial X(z)$$

$$= \frac{1}{z^2} \partial X(0) + \frac{1}{z} \partial \partial X(0) + \mathcal{O}(1) \quad (1.2.22)$$

We can construct other primary fields, that are called vertex operators:

$$V_p(z, \bar{z}) = : e^{ipX(z, \bar{z})} : \quad (1.2.23)$$

There OPEs with the stress-tensor reads:

$$T(z)V_p(0) = -\frac{1}{4} : \partial X(z) \partial X(z) : \sum_{n=0}^{\infty} \frac{(ip)^n}{n!} : X^n(0) :$$

$$= \frac{p^2}{z^2} \sum_{n=2}^{\infty} \frac{(ip)^{n-2}}{(n-2)!} : X^n(0) : + \frac{ip}{z} \sum_{n=1}^{\infty} \frac{(ip)^{n-1}}{(n-1)!} : \partial X(0)X^n(0) : + \mathcal{O}(1)$$

$$= \frac{p^2}{z^2} V_p(0) + \frac{1}{z} \partial V_p(0) + \mathcal{O}(1) \quad (1.2.24)$$

This shows that $V_p$ is a primary field of (holomorphic) dimension $p^2$. Generic states in the (holomorphic part of the ) Hilbert space of the theory will be of the form : $\partial X...\partial XV_p :$. In string theory, the parameter $p$ is interpreted as the momentum of the string, and the action of the field $\partial X$ code the (left-moving) excitations of the string.

A last and instructive computation is the OPE between the stress-tensor and itself:

$$T(z)T(0) = \frac{1}{16} : \partial X(z) \partial X(z) :: \partial X(0)\partial X(0) :$$

$$= \frac{1}{2z^4} - \frac{1}{2z^2} : \partial X(z) \partial X(0) :$$

$$= \frac{1}{2z^4} + \frac{2}{z^2} T(0) + \frac{1}{z} \partial T(0) \quad (1.2.25)$$

Notice that the stress tensor is not a primary field, because of the anomalous term in $\frac{1}{z^2}$.

### 1.2.6 The Virasoro algebra

The self-OPE of the stress-tensor has the generic form:

$$T(z)T(0) = \frac{c}{2z^4} + \frac{2}{z^2} T(0) + \frac{1}{z} \partial T(0) \quad (1.2.26)$$

8
where $c$ is called the central charge. This parameter betrays a conformal anomaly. Actually one can show that on a Riemann surface with a non-trivial curvature, the trace of the stress-tensor does not vanish:

$$Tr(T) = T_{zz} = -i\frac{c}{12}R^{(2)}$$

(1.2.27)

In string theory it is important that the total central charge of the CFT living on the worldsheet vanishes (it is the “critical dimension” condition).

From the previous computation of the conformal Ward identity we learnt that the stress tensor generates conformal transformations. We expect the Hilbert space of a CFT to be organized in representations of the conformal group. Starting from the stress-tensor we can obtain a set of generators for this group: they are the modes of the stress tensor organized in representations of the conformal group. They are the modes of the stress tensor.

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$$

(1.2.28)

or:

$$L_n = \frac{1}{2\pi i} \oint dz (z) z^{n+1}$$

(1.2.29)

We can compute the commutation relations for the $L_n$ starting from the OPE (1.2.26):

$$[L_n, L_m] = \frac{1}{(2\pi i)^2} \oint dz \oint dw T(z) T(w) z^{n+1} w^{m+1} - \frac{1}{(2\pi i)^2} \oint dz \oint dw T(z) T(w) z^{n+1} w^{m+1}$$

$$= \frac{1}{2\pi i} \oint dw \oint dz \left( \frac{c}{2} \frac{1}{(z-w)^2} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) \right) z^{n+1} w^{m+1}$$

$$= \frac{1}{2\pi i} \oint dw \left( \frac{c}{2} (n+1)n(n-1) w^{n+m-1} + 2(n+1)w^{n+m+1} T(w) + w^{n+m+2} \partial T(w) \right)$$

$$= \frac{c}{12} (n^2 - 1) \delta_{n+m,0} + (n-m) L_{n+m}$$

(1.2.30)

We obtain the Virasoro algebra. From the conformal Ward identity for primary field, we can read the action of the modes $L_n$ on a primary field $\phi$:

$$L_{n>0} \phi = 0$$

$$L_{0} \phi = \Delta \phi$$

$$L_{-1} \phi = \partial \phi$$

(1.2.31)

We also notice that any operator of the form $L_{n<0}\phi$ is not primary, since it is not annihilated by all the positive modes of the stress tensor.

### 1.2.7 Organization of the Hilbert space

From the previous discussion we understand the organization of the Hilbert space of a 2D CFT: the primary fields are lowest weight states in representations of the Virasoro algebra. All the states of the theory may be constructed by acting with the negative modes of the stress tensor on primary fields. The set of states obtained by acting in all possible way with the stress-tensor modes on a single primary state is called a Verma module.

The operator algebra takes the form:

$$\Phi_1(z, \bar{z}) \Phi_2(0,0) = \sum_{p, (k, \bar{k})} C_{12}^{p, (k, \bar{k})} z^{\Delta_p - \Delta_1 - \Delta_2 + \bar{k}} \bar{z}^{\bar{\Delta}_p - \bar{\Delta}_1 + \bar{k}} \Phi_p^{(k, \bar{k})}(0,0)$$

(1.2.32)

The sum over $p$ runs over the primary fields. The sum over $k$ and $\bar{k}$ codes the action of the negative modes of the stress tensor $T$ and $\bar{T}$ ($k$ and $\bar{k}$ are not integers but sum of integers).

Since we can compute the OPEs of the stress tensor both with itself and with the primary fields, the missing information is the OPE between primary fields.
1.2.8 Partition function

A interesting object in a 2D CFT is the partition function. It codes the conformal dimensions of all the states in the theory. In string theory, this object codes the energy of the vibration modes of the string. The partition function is defined as:

\[ Z = \text{Tr} \left( q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \right) \]  \hspace{1cm} (1.2.33)

where the trace is taken over the full Hilbert space. As explained before we can decompose this partition function as a sum over Verma modules associated with the primary fields of the theory.

Let us consider a single primary field \( \phi \). The states obtained by action of the holomorphic Virasoro modes can be expanded on the set of states defined as:

\[ L_{-k_1} L_{-k_2} ... L_{-k_n} \phi \quad 1 \leq k_1 \leq k_2 \leq ... \leq k_n \]  \hspace{1cm} (1.2.34)

Linear dependence may appear between these states. In the (generic) case where no such degeneracy appears, the partition function associated to this single module reads:

\[ Z_\phi = q^\Delta_{\phi} - c/24 \bar{q}^{\Delta_{\phi} - c/24} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)(1 - \bar{q}^n)} \]  \hspace{1cm} (1.2.35)

It can be written in terms of the Dedekind \( \eta \)-function:

\[ Z_\phi = q^\Delta_{\phi} - c/24 \bar{q}^{\Delta_{\phi} - c/24} \frac{q^{1/24} \bar{q}^{1/24}}{|\eta(q)|^2} \]  \hspace{1cm} (1.2.36)
Chapter 2

Bestiary of CFTs

In the first lecture we studied the very basic consequences of conformal invariance in two dimensions. We also encountered a first example of 2D CFT: the free boson, that describes flat directions in a string background. In this lecture we will investigate new examples of 2D CFT that are relevant for string theory.

2.1 The free fermion

The second CFT we encounter is the free complex fermion:

$$ S = \frac{1}{2\pi} \int d^2 z (\overline{\psi} \partial \psi + \overline{\psi} \partial \psi) $$

This theory is relevant when we supersymmetrize the Polyakov action in flat space, since the free fermions will be the superpartners of the free bosons. The field $\psi, \overline{\psi}$ have dimension one-half. We deduce the OPE:

$$ \psi(z)\overline{\psi}(0) = \frac{1}{z} + \mathcal{O}(1) $$

The stress-tensor is:

$$ T(z) = -\frac{1}{2} : \psi \partial \psi : $$

Thanks to Wick’s theorem it is easy to show that $\psi$ is a primary field, and that the central charge is equal to one-half. Since the action is invariant under a $Z_2$ symmetry $\psi \rightarrow -\psi$, we may consider two different periodicity conditions for the fermions. On a cylinder parameterized by $(\tau, \sigma \equiv \sigma + 2\pi)$:

$$ \psi(\sigma + 2\pi) = -\psi(\sigma) \quad (NS) \quad (2.1.4) $$

$$ \psi(\sigma + 2\pi) = +\psi(\sigma) \quad (R) \quad (2.1.5) $$

The map from the cylinder to the plane reads: $z = e^{\tau+i\sigma} = e^w$. Under this map the primary field $\psi$ transforms as:

$$ \psi_{cyl}(\sigma, \tau) \rightarrow \psi_{plane}(z) = \left( \frac{\partial z}{\partial w} \right)^{-1/2} \psi_{cyl}(z) = \frac{1}{\sqrt{z}} \psi_{cyl}(z) $$

so under this map periodic and anti-periodic boundary condition are exchanged.
In the Neveu-Schwarz sector (on the plane) the field $\psi$ may expanded as:

$$
\psi^{NS}(z) = \sum_{n \in \mathbb{Z}+1/2} \frac{\psi_n}{z^{n+1/2}}
$$

whereas in the Ramond sector we have the usual expansion:

$$
\psi^R(z) = \sum_{n \in \mathbb{Z}} \frac{\psi_n}{z^{n+1/2}}
$$

The partition function for lowest-weight representations associated to the different sectors is easily computed, and can be expressed in terms of the famous theta-functions:

$$
Tr_{NS} q^{L_0-c/24} = q^{-1/48} \prod_{n=1}^{\infty} (1 + q^{n-1/2}) = \sqrt{\frac{\theta_3}{\eta}}
$$

$$
Tr_R q^{L_0-c/24} = q^{1/24} \prod_{n=0}^{\infty} (1 + q^n) = \sqrt{\frac{\theta_2}{\eta}}
$$

We can also introduce an operator $(-1)^F$ that counts the number of fermionic generators. Inserting such an operator in the trace is equivalent to setting periodic periodicity condition in time for the fermions (as opposed to the usual choice of anti-periodic boundary conditions); in a time-ordered path integral with an even-number of fermionic fields (odd-number implies vanishing of the correlation function), making a “loop” in time with a fermionic operators amounts to make it pass through the other operators, which generates a minus sign). We obtain the partition functions:

$$
Tr_{NS} (-1)^F q^{L_0-c/24} = q^{-1/48} \prod_{n=1}^{\infty} (1 - q^{n-1/2}) = \sqrt{\frac{\theta_3}{\eta}}
$$

$$
Tr_R (-1)^F q^{L_0-c/24} = q^{1/24} \prod_{n=0}^{\infty} (1 - q^n) = \sqrt{\frac{\theta_2}{\eta}}
$$

In string theory we ask for modular invariance of the total partition function. This is possible if we add the (modulus squared of) the four previous partition functions.

### 2.2 Ghost systems

The Polyakov action is invariant under worldsheet diffeomorphism. To quantize the theory we have to gauge fix this local symmetry. The usual procedure will add a system of ghosts to the action. Their action is given by:

$$
S = \frac{1}{\pi} \int d^2zb\partial c
$$

These ghosts are anti-commuting variables. Only the $b.c$ OPE is non-vanishing:

$$
c(z)b(0) = \frac{1}{z} + O(1)
$$

The stress tensor reads:

$$
T = -2b\partial c - (\partial b)c
$$

It is an instructive exercise to show that the fields $b$ and $c$ have respective dimension 2 and $-1$, and that the central charge is $-26$. 

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In the usual superstring theory, we need to gauge fix superdiffeomorphisms. Gauge fixing of fermionic diffeomorphisms produces another system of commuting ghosts $\beta, \gamma$. The action is the same (on the torus) but the stress-tensor differs:

$$T = -\frac{3}{2} \bar{3} \partial \gamma - \frac{1}{2} (\partial \beta) \gamma$$  \hspace{1cm} (2.2.4)$$

The fields $\beta$ and $\gamma$ have respective dimension $3/2$ and $-1/2$, and that the central charge is $11$.

The total central charge for the ghosts systems of the superstring is thus equal to $-15$. This is independent of the background in which we quantize the string. To cancel this central charge, the simplest choice is to take 10 free bosons and 10 free fermions (hence the “critical dimension” of the superstring). They are of course many other choices, most of which do not have an interpretation in terms of a ten-dimensional geometry.

2.3 Minimal models

The minimal models are the “simplest” CFTs in the sense that they have only a finite number a Virasoro primary fields. They describe statistical models (Ising, etc.) at their critical points. For application to string theory we refer to the work of Seiberg and Shih.

2.4 Wess-Zumino-Novikov-Witten models

We turn now to the study of a large family of interacting CFTs. Consider a Lie group $G$. We can build a CFT with target space the group manifold of $G$. The action reads:

$$S_{WZW} = \frac{k}{16\pi} \int d^2 x Tr[-\partial^\mu g^{-1} \partial_\mu g] + k\Gamma$$  \hspace{1cm} (2.4.1)$$

where the field $g$ is a map from the worldsheet to the group $G$, and $\Gamma$ is the Wess-Zumino term:

$$\Gamma = -i \frac{k}{24\pi} \int_B d^3 y e^{\alpha \beta \gamma} Tr(g^{-1} \partial_\alpha gg^{-1} \partial_\beta gg^{-1} \partial_\gamma g)$$  \hspace{1cm} (2.4.2)$$

The three-manifold $B$ is such that its boundary is the worldsheet. One can show that the path integral is well defined if the parameter $k$ is an integer.

The theory has a global symmetry $G_L \times G_R$ acting by left- and right-multiplication on the field $g$. Let us compute the variation of the first term in the action, under an infinitesimal left-multiplication $g \rightarrow (1 + \omega)g$:

$$\delta \left( \frac{k}{16\pi} \int d^2 x Tr[-\partial^\mu g^{-1} \partial_\mu g] \right) = \frac{k}{8\pi} \int d^2 x Tr[\omega \partial^\mu (\partial_\mu gg^{-1})]$$  \hspace{1cm} (2.4.3)$$

The variation of the WZ term can be shown to be:

$$\delta \Gamma = \frac{k}{8\pi} \int d^2 x Tr[\omega \epsilon^{\mu \nu \rho} \partial_\mu (\partial_\nu gg^{-1})]$$  \hspace{1cm} (2.4.4)$$

Which gives:

$$\delta S = -\frac{1}{2\pi} \int d^2 z \bar{\partial} Tr(\omega J)$$  \hspace{1cm} (2.4.5)$$

where $J$ is the current associated to the left symmetry:

$$J(z) = -k \partial gg^{-1}$$  \hspace{1cm} (2.4.6)$$

It obeys the conservation law:

$$\bar{\partial}(\partial gg^{-1}) = 0$$  \hspace{1cm} (2.4.7)$$
Notice that $J$ is the $z$-component of a vector, and that the $\bar{z}$ component vanishes. Similarly the current associated to the right-symmetry reads:

$$J(\bar{z}) = kg^{-1}\partial g$$

(2.4.8)

In WZW models these currents play an even more important role than the stress-tensor, as we will see. The reason is that the stress-tensor can be written in terms of the currents.

Thanks to equation (2.4.5), we can compute Ward identities associated to the left-symmetry. For a generic correlation function $\langle X \rangle$ we get:

$$\langle \delta X \rangle = \langle \delta S \rangle X$$

(2.4.9)

Let us take $X = J$. Under the transformation $g \to (1 + \omega)g$ the current $J$ transforms as:

$$\delta J = -k[\partial(\omega g)g^{-1} + \partial g(-g^{-1}\omega)]$$

$$= -k\partial\omega + [\omega, J]$$

(2.4.10)

We can expand the current (and $\omega$ on a basis $t^a$ of the Lie algebra of the group $G$ : $J = J^a t_a$). Indices are contracted using the Killing metric. The generators $t^a$ satisfy:

$$[t^a, t^b] = if_{abc} t^c.$$ In components the previous equation reads:

$$\delta J^a(w) = -k\partial\omega^a(w) + if_{abc}\omega^b(w)J^c(w)$$

$$= \frac{1}{2\pi i} \oint dz \left( -\frac{k}{(z-w)^2}\omega(z) + \frac{1}{z-w}if_{bc}\omega^b(z)J^c(z) \right)$$

(2.4.11)

We obtain for the Ward identity:

$$\left\langle \frac{1}{2\pi i} \oint dz\omega^b(z) \left( -\frac{kK_{ab}}{(z-w)^2} + \frac{1}{z-w}iK_{bc}\omega^c(z) \right) \right\rangle = \left\langle -\frac{1}{2\pi i} \oint d^2\omega^b(z)J^b(z) \right\rangle$$

(2.4.12)

We deduce the OPE between two components of the current:

$$J^a(z)J^b(w) \sim \frac{kK_{ab}}{(z-w)^2} + \frac{iK_{ab}}{z-w}J^c(w) z^{-w}.$$$$J^a(z)J^b(w) \sim \frac{kK_{ab}}{(z-w)^2} + \frac{iK_{ab}}{z-w}J^c(w) z^{-w}.$$ 

(2.4.13)

Expanding the current in modes, this OPE implies the following commutation relations:

$$[J^a_n, J^b_m] = knK_{ab}\delta_{n+m,0} + i f_{abc} J^c_{n+m}$$

(2.4.14)

They define an affine Lie algebra at level $k$.

The stress tensor can be written as a bilinear in the currents. This is called the Sugawara construction:

$$T = \frac{1}{2(k + \bar{h})} : J^a J_a :$$

(2.4.15)

where $\bar{h}$ is the dual Coxeter number of the Lie algebra. It is related to the Killing metric as:

$$f^{abc} f_{dc} = 2h_{k}.$$$$f^{abc} f_{dc} = 2h_{k}.$$ 

(2.4.16)

A convenient way to define the normal ordering prescription in the definition of $T$ is the following:

$$: J^a J_a : (z) = \frac{1}{2\pi i} \oint dw \frac{1}{w-z} J^a(w) J_a(z)$$

(2.4.17)

The OPE (2.4.13) implies that the current is a primary field of dimension one:

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}$$

(2.4.18)
This in turn implies the canonical OPE for the stress tensor:
\[ T(z)T(w) = \frac{k \dim G}{2(k + \hbar)(z - w)^2} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w} \] (2.4.19)

We can read the central charge:
\[ c = \frac{k \dim G}{k + \hbar} \] (2.4.20)

These models give a nice example in which the Virasoro algebra is embedded into an even bigger symmetry algebra: an affine Lie algebra. These models have an infinite number of Virasoro primary fields, which renders difficult to solve the model using only the Virasoro symmetry. The affine Lie algebra on the other hand has much less primary fields (i.e., highest-weight states of representations), and allows for a complete solution of the model.
Chapter 3

Non-trivial worldsheets

Until now we discussed 2D CFT defined on a plane. In this context the holomorphic and anti-holomorphic sides of the theory decouple. In this lecture we will consider less trivial worldsheets. First we will discuss 2D CFT defined on a torus. The end of the lecture will be about worldsheets with boundaries.

3.1 2D CFT on the torus

N-loops amplitudes in string theory are obtained by computing correlation functions on Riemann surfaces of genus $N$. CFTs well-defined on such worldsheet must satisfy some constraints. We will illustrate this fact in the case of the torus.

A torus is defined by a lattice of periods $w_{1,2}$:

$$ z \equiv z + w_1 , \quad z \equiv z + w_2 \quad (3.1.1) $$

The periods $w_{1,2}$ define an unit cell of the lattice. The same lattice can be defined by other periods $w'_{1,2}$ as long as both pairs are related by an $SL(2, \mathbb{Z})$ transformation (they must be integer combinations one of the other, and the volume must be preserved). We can choose the coordinates so that $w_1$ is real (and equal to one). A torus is defined by the modular parameter $\tau$:

$$ \tau = \frac{w_2}{w_1} \quad (3.1.2) $$

The group $SL(2, \mathbb{Z})$ acts on the modular parameter $\tau$ as:

$$ \tau \rightarrow \frac{a \tau + b}{c \tau + d} \quad (3.1.3) $$

Since changing the sign of $a, b, c, d$ does not change the result, the modular group is actually $PSL(2, \mathbb{Z})$. One can show that it is generated by the transformations $T$ and $S$:

$$ T : \tau \rightarrow \tau + 1 \quad ; \quad S : \tau \rightarrow -\frac{1}{\tau} \quad (3.1.4) $$

The partition function on the torus is defined as:

$$ Z = Tr e^{2\pi i \tau (L_0 - c/24)} e^{-2\pi i \bar{\tau} (L_0 - c/24)} \quad (3.1.5) $$

It must be invariant under modular transformations for the theory to be well-defined.

The spectrum of the theory is organized in representations of the Virasoro algebra (or possibly an even bigger symmetry algebra), so the partition function can be written as a sum of product of a holomorphic characters with an anti-holomorphic one. The pairing must be chosen wisely so that the result is modular invariant. Modular invariance under
the $T$-transformation is quite easily achieved: it is sufficient that the conformal dimensions in the holomorphic and the anti-holomorphic characters differ by integers. Invariance under the $S$-transformation is more subtle. Generically we can write its action on a character as:

$$S : \chi_i(q) \rightarrow \chi_i(\tilde{q}) = \sum_j S_{ij} \chi_j(q)$$

(3.1.6)

where the sum over $j$ is performed over the highest-weight representations. In many theories the characters transform in unitary representations of the modular group, i.e. the matrix $S_{ij}$ is unitary. Thus the diagonal partition function is modular invariant:

$$Z_{\text{diag}} = \sum_i |\chi_i(q)|^2$$

(3.1.7)

### 3.2 Worldsheets with boundaries

The motivation to consider such worldsheets comes from the study of D-branes. The elementary degrees of freedom of a D-brane are the open strings that live on its worldvolume. The dynamics of these open strings is coded by the 2D CFT that lives on their worldsheets. The ends of the open string are attached to the D-brane, thus in the worldsheet language a D-brane is a boundary condition. Operators that create open strings states on the worldsheet are operators that may change boundary conditions.

The relevant object in a 2D CFT that encodes boundary conditions is called a boundary state. We will introduce these objects. For more details see for example Schomerus’ lecture notes, “Lectures on branes in curved backgrounds”.

Consider a 2D CFT. We call $\mathcal{W}$ the maximal chiral algebra realized in this theory. Generically $\mathcal{W}$ will be the Virasoro algebra, but it may be bigger (e.g. an affine Lie algebra in WZW models, or a supersymmetric extension of the Virasoro algebra in the context of superstrings). The Hilbert space of the theory will be organized in representations $i$ of the chiral algebra $\mathcal{W}$. To each one of these representations we associate a character $\chi_i$:

$$\chi_i(q) = Tr_i \left( q^{L_0 - c/24} \right)$$

(3.2.1)

#### 3.2.1 Boundary conditions

Let us consider a CFT defined on the upper-half plane (same topology as a disc). We have to impose boundary conditions on the boundary, i.e. on the real line. First we want to forbid any energy flow across the boundary. This requires:

$$T(z) = \bar{T}(\bar{z})|_{z = \bar{z}}$$

(3.2.2)

We also want boundary conditions to preserve the $\mathcal{W}$ symmetry. We need an automorphism $\Omega$ of the algebra $\mathcal{W}$. Then for every generator $w$ of the symmetry algebra we set:

$$w(z) = \Omega \bar{w}(\bar{z})|_{z = \bar{z}}$$

(3.2.3)

#### 3.2.2 Boundary states

Let us introduce the concept of boundary state. Suppose we work on the upper half plane with boundary conditions denoted as $\alpha$. We introduce a (formal) state $|\alpha\rangle$ satisfying:

$$\langle \phi_1(z_1)\ldots\phi_p(z_p) |\alpha\rangle = \langle \alpha | \phi_1(z_1)\ldots\phi_p(z_p)|0\rangle$$

(3.2.4)

On the left-hand side we have a partition function computed on the upper-half plane with boundary condition $\alpha$. On the right-hand side we have a correlation function computed on the whole plane but with a non-trivial asymptotic state: the boundary state $|\alpha\rangle$. 

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Boundary states must satisfy the constraints \( \text{FRMQMQG} \) and \( \text{FRMRGM} \). There exist one solution \( \langle i | \rangle \) to these constraints for each representation \( i \) of the symmetry algebra \( \mathcal{W} \). They are called Ishibashi states. They satisfy:

\[
\langle (j | q^{L_0-c/24} | i \rangle \rangle = \delta_{i,j} \chi_i(q)
\]

Ishibashi states give a basis of the boundary states.

### 3.2.3 Cardy’s construction of D-branes

D-branes are in one-to-one correspondence with the boundary conditions we impose on the boundary of the open string worldsheet. A D-brane will be written as a linear combination of Ishibashi states:

\[
|D\rangle = \sum_i A_i |i\rangle \quad (3.2.6)
\]

The choice of the coefficients \( A_i \) is not free. The main constraint is known as the Cardy condition. To understand this constraint let us work on a ring, with boundary condition \( D_1 \) et \( D_2 \). The partition function then reads:

\[
\langle D_1 | q^{L_0-c/24} | D_2 \rangle = \sum_i A_i^{D_1} A_i^{D_2} \chi_i(q) \quad (3.2.7)
\]

The previous quantity can be thought of as the tree-level scattering amplitude for a close string from the D-brane \( D_1 \) to the D-brane \( D_2 \). In order to interpret this quantity in an open-string language we perform a modular transformation \( S \). We obtain the one-loop partition function for an open string stretching between the D-branes \( D_1 \) et \( D_2 \):

\[
Z_{D_1-D_2} (\bar{q}) = \sum_{i,j} A_i^{D_1} A_i^{D_2} S_{ii} \chi_i (\bar{q}) \quad (3.2.8)
\]

For this partition function to make sense we need the coefficients multiplying the characters to be positive integers: this is the Cardy condition. The coefficients of the \( S \)-matrix are imaginary, so this condition is difficult to satisfy. Happily there is a generic formula which tells us how to construct positive integers with the elements of the \( S \)-matrix (in unitary rational CFTs). This is the Verlinde formula:

\[
\sum_i S_{il} S_{lj} S_{kl}^* = N_{ik} \quad (3.2.9)
\]

Here \( N_{ik}^j \) are positive integer called fusion rules. They give the number of independent couplings between the representations \( i, j \) and \( k \) in the theory.

We deduce that we can construct one consistent D-brane for each representation of the symmetry algebra \( \mathcal{W} \). The boundary state reads:

\[
|D_i\rangle = \sum_j \frac{S_{ij}}{\sqrt{S_{0j}}} |j\rangle \quad (3.2.10)
\]

The coefficients appearing in this linear combination code the coupling between the D-brane and the close string states. More precisely they give the one-point function for the different string states, and thus encode the backreaction of the D-brane on the background.

### 3.2.4 An example: D-brane in the free boson CFT

In order to make the previous abstract discussion a little bit more concrete we consider a simple example: the free boson. We have seen that we can construct a consistent D-brane starting from an automorphism of the chiral symmetry algebra (cf \( 3.2.3 \)), and
a representation of this algebra. In the case of the free boson we have an affine $u(1)$ Lie algebra, with chiral generator $\partial X$. The stress tensor is given by the Sugawara construction: $T \propto \partial X \partial X$. Since we must have $T = \bar{T}|_{z=\bar{z}}$ on the boundary, the only allowed gluing conditions for the currents are:

$$\partial X = \pm \partial X|_{z=\bar{z}}$$

(3.2.11)

They are the well-known Neumam and Dirichlet boundary conditions. Once we pick one of these, each representation of the $u(1)$ Lie algebra gives a D-brane. These representations are labelled by a continuous parameter. In the case of Dirichlet boundary conditions, this parameter is interpreted as the position of the D-brane. Then we know from T-duality that in the case of Neuman boundary condition this parameter becomes a Wilson line.
Chapter 4

Supersymmetric CFTs

CFTs give the vacua of string theory. However in most of these vacua we find tachyonic excitations of the string. In this last lecture we will present some techniques used to construct stable vacua of string theory. In a landscape picture, a random CFT gives a saddle point; we will now find local minima.

4.1 Orbifolds

The orbifold procedure is a convenient way to generate new CFTs starting from a CFT with a finite symmetry group $\Gamma$. First we project out all the states that are not invariant under the symmetry $\Gamma$. The resulting theory is incomplete: modular invariance for example is generically lost. The second step is to add new operators to the theory, satisfying twisted boundary conditions:

$$\phi(z + 1) = \gamma_1 \phi(z), \quad \phi(z + \tau) = \gamma_2 \phi(z) \quad (4.1.1)$$

where $\gamma_1$ and $\gamma_2$ are commuting elements of the group $\Gamma$. The partition function of the new theory reads:

$$Z = \sum_{(\gamma_1, \gamma_2) \in \Gamma^2} Z_{\gamma_1, \gamma_2} \quad (4.1.2)$$

Where $Z_{\gamma_1, \gamma_2}$ is the partition function of the sector twisted by the elements $\gamma_1$ and $\gamma_2$. The partition function (4.1.2) is modular invariant. Indeed let us investigate the action of modular transformations on the elementary partition functions $Z_{\gamma_1, \gamma_2}$. Under a $T$ transformation, we obtain fields satisfying:

$$\phi(z + 1) = \gamma_1 \phi(z), \quad \phi(z + \tau + 1) = \gamma_2 \phi(z) \quad (4.1.3)$$

which implies:

$$TZ_{\gamma_1, \gamma_2} = Z_{\gamma_1, \gamma_1 \gamma_2} \quad (4.1.4)$$

Under a $S$ transformation, we obtain:

$$\phi(z + 1) = \gamma_1 \phi(z), \quad \phi(z - 1/\tau) = \gamma_2 \phi(z) \quad (4.1.5)$$

which implies:

$$SZ_{\gamma_1, \gamma_2} = Z_{-\gamma_2, \gamma_1} \quad (4.1.6)$$
4.2 \( N = 1 \) SuperConformal Algebra

The basic idea to get rid of tachyons in string theory is to implement SUSY in spacetime (Some non-SUSY stable vacua of string theory are known but they are very difficult to construct or/and very far away from phenomenology). As a first step in this direction we will first implement SUSY on the worldsheet. The minimal supersymmetric extension of the Virasoro algebra in two dimensions is called the \( \mathcal{N} = 1 \) superconformal algebra. It is generated by the stress-tensor \( T \) and a supercurrent \( G \):

\[
G(z)G(w) = \frac{2c}{3(z-w)^3} + \frac{2}{z-w} + ...
\]

\[
T(z)G(w) = \frac{3}{2(z-w)^2} + \frac{\partial G(w)}{z-w} + ...
\]

\[
T(z)T(w) = \frac{3}{2(z-w)^2} + \frac{2}{(z-w)^2} \frac{\partial T(w)}{z-w} + ...
\]  \( \text{(4.2.1)} \)

4.3 \( N = 2 \) SuperConformal Algebra

We can extend the \( \mathcal{N} = 1 \) SCA in a \( \mathcal{N} = 2 \) SCA. This algebra is generated by two supercurrents \( G^\pm \) and a bosonic \( U(1) \) current \( J \), in addition to the stress-tensor \( T \):

\[
J(z)J(w) = \frac{c}{3(z-w)^2} + ...
\]

\[
J(z)G^\pm(w) = \pm \frac{G^\pm(w)}{z-w} + ...
\]

\[
T(z)J(w) = \frac{J(w)}{z-w} + \frac{\partial J(w)}{z-w} + ...
\]

\[
G^+(z)G^-(w) = \frac{2c}{3(z-w)^3} + \frac{2}{z-w} + \frac{J(w)}{z-w} + \frac{\partial J(w)}{z-w} + 2 \frac{T(w)}{z-w} + ...
\]

\[
T(z)G^\pm(w) = \frac{3}{2(z-w)^2} + \frac{\partial G^\pm(w)}{z-w} + ...
\]  \( \text{(4.3.1)} \)

We can transform these OPE’s in (anti-)commutation relations for the modes of the generators. Of course the moding (integer or half integer) of the fermionic generators will depend on whether we are in the NS or in the R sector. Let us consider for example the following anti-commutation relation (in the NS sector):

\[
\{G_{1/2}^+, G_{-1/2}^-\} = \frac{1}{(2\pi i)^2} \int_0^\infty dz \int_0^\infty dw G^+(z)G^-(w)w^0 + \int_0^\infty dw \int_0^\infty dz G^-(w)w^0 G^+(z)z^1
\]

\[
= \frac{1}{(2\pi i)^2} \int_0^\infty dw \int_0^\infty dz \left( \frac{2c}{3(z-w)^2} + \frac{2J(w)}{z-w} + \frac{\partial J(w)}{z-w} + \frac{T(w)}{z-w} + ...ight)
\]

\[
= \frac{1}{2\pi i} \int_0^\infty dw \left( 0 + 2J(w) + w\partial J(w) + 2wT(w) \right)
\]

\[
= \frac{1}{2\pi i} \int_0^\infty dw \left( 2 \sum_n \frac{J_n}{w^{n+1}} + \sum_n (-n-1) \frac{J_n}{w^{n+2}} + \sum_n \frac{L_n}{w^{n+2}} \right)
\]

\[
= J_0 + 2L_0
\]  \( \text{(4.3.2)} \)

Similarly one can show that \( \{G_{1/2}^+, G_{1/2}^+\} = -J_0 + 2L_0 \). In an unitary theory, these equations imply for any operator:

\[
2\Delta \geq |Q|
\]  \( \text{(4.3.3)} \)
where $\Delta$ is the conformal dimension and $Q$ is the $U(1)$ charge. We will come back to this inequality in the next section.

### 4.4 GSO projection

The realization of SUSY on the worldsheet does not imply that the spacetime string theory is also supersymmetric. However the realization of the $N = 2$ SCA on the worldsheet allow for the construction of a stable vacuum of string theory. This procedure is called the (generalized) GSO projection.

We assume that the $U(1)$ charges of the operators of the theory belong to $\frac{1}{2}Z$ with $k$ an integer. We isolate a discrete subgroup $Z_k$ of the $U(1)$ symmetry that acts on the operators by a multiplication by $e^{2\pi iQ}$ where $Q$ is the $U(1)$ charge of the operator. The first step of the GSO projection is to perform an orbifold of the theory by this group $Z_k$. The resulting orbifold theory has only integer charges.

In the second step we project out the states in the NS sector with even $U(1)$ charge (once we take into account the superghosts this amounts to an additional $Z_2$ orbifold). We have to perform a similar projection in the R sector, but the choice of charge parity for the state we project is free. This gives rise to two stable vacua: one in type IIB if the parity we project is the same in the holomorphic and anti-holomorphic R sectors, and one in type IIA in the other case.

We will now show that the resulting theory is free of tachyons (in the NS sector). Remember that the mass of a string state is given by:

$$M^2 = \Delta - \frac{1}{2}$$

where $\Delta$ is the conformal dimension of the corresponding worldsheet operator. In our GSO-projected theory, all the states in the NS sector have odd $U(1)$ charge, thus $|Q| \geq 1$. But according to equation (4.3.3) this implies that $\Delta \geq 1/2$. So $M^2 \geq 0$.

An additional symmetry of the $N = 2$ SCA called spectral flow induce a one-to-one correspondence between the states in the NS and in the R sector, allowing to extend the proof to the R sector.

### 4.5 Beyond the RNS string

The previous strategy to obtain vacua of string theory applies in the traditional “RNS” string. Unfortunately this formalism does not apply in spacetime with RR fluxes. As we have seen the relationship between worldsheet and spacetime SUSY in this formalism is not straightforward. Roughly the reason is that supersymmetrization of the bosonic degrees of freedom on the worldsheet produces worldsheet fermions with bosonic spacetime indices. What we would like to get explicit SUSY in spacetime, and to be able to write down worldsheet couplings to RR fields, are worldsheet fields with spacetime indices. This was realized in the early days of string theory. The first attempt is called the Green-Schwarz string. It consists of a sigma model on a superspace with coordinates $X^\mu, \theta^a$. This formalism is difficult to quantize. A more promising approach has been developed by Berkovits et al. (“pure spinors”, “hybrid strings”, etc). In these approaches spacetime fermionic coordinates are represented by $(b, c)$ ghosts systems (with spacetime fermionic indices, traditionally they are called $(\theta^a, d_a)$) on the worldsheet. The sigma models reads (roughly):

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left[ \frac{1}{2} (G_{MN}(Z) + B_{MN}(Z)) \partial Z^M \partial Z^N + P^{\alpha\beta}(Z) d_\alpha \bar{d}_\beta + ... \right]$$

(4.5.1)
Then spacetime SUSY is manifest (but the worldsheet theory is not supersymmetric). The RR couplings are coded in $P^{\alpha\dot{\beta}}$ as:

$$
P^{\alpha\dot{\beta}}|_{\theta=\dot{\theta}=0} = -\frac{i}{4} e^\phi F_{RR}^{\alpha\dot{\beta}}$$  \hspace{1cm} (4.5.2)

where $F_{RR}^{\alpha\dot{\beta}}$ is the RR field strength in bispinor notation.