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Volume 1

Higher Spin Gauge Theories

PROCEEDINGS

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Foreword

The International Institutes for Physics and Chemistry, founded by Ernest Solvay, are an independent non-profit institution whose mission is to support first class fundamental research at the international level in physics, chemistry and related areas. Besides the renowned Solvay conferences, the Institutes have launched in 2004 a workshop program on topical questions at the frontiers of science.

The first workshop of this new series took place in Brussels from May 12 through May 14, 2004 and was devoted to the question of higher spin gauge fields. This volume contains the lectures given at the workshop. The contributions were prepared with the assistance of younger colleagues (Ph.D. students and postdocs), which gives them a unique pedagogical value. I am very grateful to the speakers and all the authors for the thoughtful work and care that went into this volume, which constitutes an excellent introduction to the subject, bringing the reader from the basics of the field to its frontiers. I am convinced that this volume will have a long lasting impact and will become a "must" to all researchers interested in the subject.

Marc Henneaux
Director of the International Solvay Institutes

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“Massive” Higher Spin Multiplets and Holography

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Abstract. We review the extrapolation of the single-particle string spectrum on $AdS_5 \times S^5$ to the Higher Spin enhancement point and the successful comparison of the resulting spectrum with the one of single-trace gauge-invariant operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We also describe how to decompose the common spectrum in terms of massless and massive representations of the relevant Higher Spin symmetry group.

Based on the lecture delivered by M. Bianchi at the First Solvay Conference on Higher-Spin Gauge Theories held in Bruxelles, on May 12-14, 2004.

1 Introduction

We present an overview of the work done by one of the authors (M.B.) in collaboration with N. Beisert, J.F. Morales and H. Samtleben [1, 2, 3][1]. After giving some historical motivations for the interest in higher spin (HS) gauge fields and currents, we very briefly and schematically review some of the achievements of the holographic AdS/CFT correspondence[2]. We mostly but not exclusively focus on protected observables that do not change as we vary 't Hooft coupling constant $\lambda = g^2_{_{YM}} N$. We then discuss how the single-particle string spectrum on $AdS_5 \times S^5$ can be extrapolated to the HS enhancement point [9, 10, 11, 12, 13, 14] and how it can be successfully compared with the spectrum of single-trace gauge-invariant operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory [1,2]. To achieve the goal we rely on the aid of Polya theory [15]. We also decompose the resulting spectrum in terms of massless and massive representations of the relevant HS symmetry group [3]. Eventually, we concentrate our attention on the generalization of the HS current multiplets, *i.e.* semishort multiplets, which saturate a unitary bound at the HS enhancement point and group into long ones as we turn on interactions. Finally, draw our conclusions and perspectives for future work. Properties of HS gauge theories are extensively covered by other contributions to this conference [16,17,18,19,20] as well as the reviews *e.g.* [21,22,23].

2 Historical motivations

The physical interest in HS currents dates back to the studies of QCD processes, such as deep inelastic scattering, where the structure of hadrons was probed by electrons or neutrinos. The process is studied at a scale $Q^2 = -q^2$, related to the momentum q transferred by the photon, which is much larger than the typical mass parameter of the theory Λ_{QCD}.

The fraction of momentum carried by the struck "parton", *i.e.* one of the hadron's constituents, is given by the Bjorken variable

$$0 \leq \xi = x_B = \frac{Q^2}{2P \cdot q} \leq 1 \,, \tag{1}$$

where P is the momentum of the hadron. Note that ξ is kinematically fixed. The optical theorem relates the amplitude of the process to the forward Compton amplitude $W^{\mu\nu}$

$$W^{\mu\nu} = W_1(x_B)\Big(\frac{q^\mu q^\nu}{q^2} - \eta^{\mu\nu}\Big) + W_2(x_B)\Big(P^\mu - q^\mu \frac{P \cdot q}{q^2}\Big)\Big(P^\nu - q^\nu \frac{P \cdot q}{q^2}\Big) =$$

$$= i \int d^4x e^{iqx} \int_0^1 \frac{d\xi}{\xi} \sum_i f_i(\xi) \langle q_i(\xi P)| T(J^\mu(x) J^\nu(0)) |q_i(\xi P)\rangle \,, \tag{2}$$

where $W_1(x)$ and $W_2(x)$ are scalar structure functions and $f_i(\xi)$ are the parton distribution functions, which depend on non-perturbative dynamical effects such as confinement. Parity-violating terms which can appear in weak interactions are omitted for simplicity. For non interacting spin 1/2 partons, the structure functions satisfy Callan-Gross relations

$$\mathcal{I}m W_1 = \pi \sum_i e_i^2 f_i \,, \qquad \mathcal{I}m W_2 = \frac{4x_B}{Q^2} \mathcal{I}m W_1 \,. \tag{3}$$

[1]A shorter account can be found in [4]

[2]For recent reviews see *e.g.* [5, 6, 7, 8].

Operator Product Expansion (OPE) yields

$$W^{\mu\nu} = \sum_i e_i^2 \Big[\sum_{M=0}^{\infty} \frac{P^\mu P^\nu}{Q^2} \Big(\frac{2P\cdot q}{Q^2}\Big)^{M-2} A_i^{(n)}(Q^2) - \frac{1}{4}\eta^{\mu\nu} \sum_{M=0}^{\infty} \Big(\frac{2P\cdot q}{Q^2}\Big)^{M} A_i^{(n)}(Q^2) \Big] + \dots\,, \quad (4)$$

where the dominant contribution arises from operators with lowest twist $\tau = \Delta - s = 2 + \dots$ The non-perturbative information is coded in the coefficients $A_i^{(n)}$ which can be related to the matrix elements in the hadronic state of totally symmetric and traceless HS currents built out of quark fields ψ

$$A_i^{(n)}: \qquad \langle P|\bar{\psi}_i \gamma^{(\mu_1} D^{\mu_2} \dots D^{\mu_n)} \psi_i |P\rangle \quad (5)$$

Higher twist operators and flavour non-singlet operators, such as

$$\eta_{\lambda_i\lambda_j} \dots \eta_{\tau_\nu\rho_\tau} \dots \bar{\psi}_i D^{\mu_1} \dots D^{\mu_k} F^{\rho_1\nu_1} D^{\lambda_1} \dots D^{\lambda_\rho} F^{\rho_2\nu_2} \dots D^{\tau_1} \dots D^{\tau_\nu} \psi_j\,,$$

can appear in non-diagonal OPE's and produce mixing with purely gluonic operators. Their contribution to the OPE of two currents is suppressed at large Q^2.

2.1 Broken scale invariance

It is well known that QCD is only approximately scale invariant in the far UV regime of very large Q^2 where it exposes asymptotic freedom [24]. Scale invariance is indeed broken by quantum effects, such as vacuum polarization that yields $\beta \neq 0$, and dimensional transmutation generates the QCD scale

$$\Lambda_{QCD} = \mu e^{-\frac{8\pi^2}{bg^2(\mu)}}\,. \quad (6)$$

The coefficient functions $A_i^{(n)}(Q^2)$ turn out to be Mellin transforms of the parton distributions and evolve with the scale Q^2 due to quantum effects. Depending on the parity of n, the parton and antiparton distributions contribute with the same or opposite sign. Defining $f_i^\pm = f_i \pm \bar{f}_i$, one has

$$A_i^{(n)}(Q^2) = \int_0^1 d\xi \xi^{n-1} f_i^+(\xi, Q^2) \quad (7)$$

in the case of even n and similarly

$$A_i^{(n)} = \int_0^1 d\xi \xi^{n-1} f_i^-(\xi, Q^2)\,, \quad (8)$$

in the case of odd n.

Parton distributions satisfy sum rules arising from global conservation laws stating *e.g.* that the net numbers of constituents of a given hadron do not depend on the scale

$$\int_0^1 d\xi f_i^-(\xi, Q^2) = n_i\,, \quad (9)$$

where n_i is independent of Q^2. For instance we know that protons are made of two up quarks $n_u = 2$ and one down quark $n_d = 1$ at each scale. Similarly the total momentum, including the gluonic contribution labelled by g, should be equal to the momentum of the hadron, and one obtains another sum rule of the form

$$\sum_i \langle x_i \rangle + \langle x_g \rangle = 1 \quad (10)$$

The evolution of the parton distributions is governed by Altarelli-Parisi (AP) equations [25][3]. For odd n, there is no operator mixing and the GLAP equations are "diagonal"

$$\frac{d}{dt}A_i^{(n)}(t) = \frac{\alpha_s(t)}{2\pi}\hat{P}^{(n)}A_i^{(n)}(t)\,, \quad t = \log Q^2 \quad , \tag{11}$$

To lowest order, the relevant kernel is

$$\hat{P}^{(n)} = \int_0^1 dz z^{n-1} P_{q\leftarrow q}(z) = \int_0^1 dz z^{n-1}\Big(\frac{4}{3}\Big)\Big[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\Big]\,. \tag{12}$$

that can be identified with the Mellin transform of the probability for a spin 1/2 parton to emit an almost collinear gluon. The result turns out to be simply given by

$$\hat{P}^{(n)} = -\frac{2}{3}\Big[1 + 4\sum_{k=2}^{n}\frac{1}{k} - \frac{2}{n(n+1)}\Big] \quad . \tag{13}$$

The presence of the harmonic numbers calls for a deeper, possibly number theoretic, interpretation and implies that the anomalous dimensions γ_S of HS currents at one loop behave as $\gamma_S \sim \log S$ for $S \gg 1$. Remarkably enough, the same leading behaviour holds true at two and higher loops [27].

For even n there is mixing with purely gluonic HS currents of the form

$$J_v^{(\mu_1\cdots\mu_n)|} = F_\lambda^{(\mu_1}D^{\mu_2}\ldots D^{\mu_{n-1}}F^{\mu_n)|\lambda} \quad , \tag{14}$$

In the holographic perspective twist two HS currents should correspond to nearly massless HS gauge fields in the bulk theory. Moreover, in $\mathcal{N}=4$ SYM one has to take into account twist two currents that are made of scalars and derivatives thereof

$$J_\phi^{(\mu_1\cdots\mu_n)|} = \varphi_i D^{\mu_1}\ldots D^{\mu_n}\varphi^i \quad . \tag{15}$$

Although $\mathcal{N}=4$ SYM theory is an exact superconformal field theory (SCFT) even at the quantum level, thanks to the absence of UV divergences that guarantees the vanishing of the β-function, composite operators can have nonvanishing anomalous dimensions.

2.2 Anomalous dimensions

We now turn to discuss anomalous dimensions and unitary bounds. In a CFT$_D$ a spin S current with scaling dimension $\Delta = S + D - 2$ is necessarily conserved. For instance, a vector current with $S = 1$ and $\Delta = 3$ in $D = 4$ has a unique conformal invariant 2-point function of the form

$$\langle J^\mu(x)J^\nu(0)\rangle = (\partial^\mu\partial^\nu - \partial^2\delta^{\mu\nu})\frac{1}{x^4}\,, \qquad (D=4) \quad , \tag{16}$$

that implies its conservation. For $\Delta = 3 + \gamma$ one finds instead

$$\langle \hat{J}^\mu(x)\hat{J}^\nu(0)\rangle = \frac{1}{x^{2\gamma}}(\partial^\mu\partial^\nu - \partial^2\delta^{\mu\nu})\frac{1}{x^4} \quad , \tag{17}$$

that leads to the (anomalous) violation of the current. Anomalous dimensions of HS currents satisfy positivity constraints. For instance, in $D = 4$, a scaling operator carrying non-vanishing Lorentz spins j_L and j_R satisfies a unitary bound of the form

$$\Delta \geq 2 + j_L + j_R \tag{18}$$

[3]Closely related equations were found by Gribov and Lipatov for QED [26].

At the threshold null states of the form $A = \partial^\mu J_\mu$ (dis)appear. When $j_L = 0$ or $j_R = 0$, *e.g.* for spin 1/2 fermions and scalars, the unitary bound (18) takes a slightly different form

$$\Delta \geq 1 + j \quad . \tag{19}$$

The identity is the only (trivial) operator with vanishing scaling dimension.

In $\mathcal{N} = 4$ SYM the situation gets a little bit more involved [28, 29, 30, 31]. The highest weight state (HWS) of a unitary irreducible representation (UIR) of $(P)SU(2,2|4) \subset U(2,2|4) = SU(2,2|4) \times U(1)_B$ can be labelled by

$$\mathcal{D}\Big(\Delta, (j_L, j_R), [q_1, p, q_2]; B, C; L, P\Big) , \tag{20}$$

where Δ is the dimension, (j_L, j_R) are the Lorentz spins, $[q_1, p, q_2]$ are the Dynkin labels of an $SU(4)$ R-symmetry representation. The central charge C, which commutes with all the remaining generators but can appear in the anticommutator of the supercharges, and the "bonus" $U(1)_B$ charge, related to an external automorphism of $SU(2,2|4)$, play a subtle role in the HS generalization of the superconformal group. The discrete quantum number P can be associated with the transposition of the gauge group generators or with a generalized world-sheet parity of the type IIB superstring. Finally the length L of an operator or a string state which is related to twist, but does not exactly coincide with it, is a good quantum number up to order one loop in λ.

Setting $C = 0$ and neglecting the $U(1)_B$ charge, there are three types of UIR representations of $PSU(2,2|4)$ relevant for the description of $\mathcal{N} = 4$ SYM theory.

- type A

  For generic (j_L, j_R) and $[q_1, p, q_2]$, one has

  $$\Delta \geq 2 + j_L + j_R + q_1 + p + q_2 \quad , \tag{21}$$

  that generalizes the unitary bound of the conformal group. The bound is saturated by the HWS's of "semishort" multiplets of several different kinds. Current-type multiplets correspond to $j_L = j_R = S/2$ and $p = q_1 = q_2 = 0$. Kaluza-Klein (KK) excitations of order p to $j_L = j_R = 0$ and $q_1 = q_2 = 0$. Above the bound, multiplets are long and comprise 2^16 components times the dimension of the HWS.

- type B

  For $j_L j_R = 0$, say $j_L = 0$ and $j_R = j$, one has

  $$\Delta \geq 1 + j_R + \frac{1}{2} q_1 + p \quad , \tag{22}$$

  that generalizes the unitary bound of the conformal group. At threshold one finds 1/8 BPS multiplets.

- type C

  For $j_L = j_R = 0$ and $q_1 = q_2 = q$, one has

  $$\Delta = 2q + p \quad , \tag{23}$$

  the resulting UIR is 1/4 BPS if $q \neq 0$ and 1/2 BPS when $q = 0$. In the 1/2 BPS case, the number of components is $2^8 p^2 (p^2 - 1)/12$, the multiplet is protected against quantum corrections and is ultrashort for $p \leq 3$. For $p = 1$ one has the singleton, with 8 bosonic

and as many fermionic components, that corresponds to the elementary (abelian) vector multiplet. In the 1/4 BPS case, if the HWS remains a primary when interactions are turned on, the multiplet remains 1/4 BPS and short and protected against quantum corrections. For single trace operators, however, the HWS's all become superdescendants and acquire anomalous dimensions in a pantagruelic Higgs-like mechanism that deserves to be called *"La Grande Bouffe"*.

3 Lessons from AdS/CFT

Before entering the main part of the lecture, it may be useful to summarize what we have learned from the holographic AdS/CFT correspondence [5, 6, 7, 8]. Let us list some of the important lessons.

- The spectrum of 1/2 BPS single-trace gauge-invariant operators at large N matches perfectly well with the Kaluza-Klein spectrum of type IIB supergravity on S^5.

- The 3-point functions of chiral primary operators (CPO's) $\mathcal{Q}_p$, which are HWS's of 1/2 BPS multiplets,

$$\langle \mathcal{Q}_{p_1}(x_1)\mathcal{Q}_{p_2}(x_2)\mathcal{Q}_{p_3}(x_3)\rangle = C(p_1,p_2,p_3;N)\prod_{i<j} x_{ij}^{-2(l_i+l_j-\Sigma)}\,, \tag{24}$$

  are not renormalized by interactions and, as shown in [32], only depend on the quantum numbers p_i, associated with the spherical harmonics on S^5, and on the number of colors N, but neither on the gauge coupling g_{YM} nor on the vacuum angle ϑ_{YM} [33].

- There are some additional observables that are not renormalized. In particular, extremal and next-to extremal n-point correlators of CPO's are exactly the same as in the free theory [34]. A correlator of CPO's is (next-to) extremal when p_0 is the sum (minus two) of the remaining p_i and one finds

$$\langle Q_{l_0}(x_0)Q_{l_1}(x_1)\dots Q_{l_n}(x_n)\rangle = G^{free}_{n+1}\,, \qquad l_0=\sum_{i=1}^{n} l_i(-2)\,. \tag{25}$$

- For near extremal correlators with $l_0=\sum_i l_i(-4,-6,\dots)$ one has partial non-renormalization [35]. In practise these correlation functions depend on lesser structures than naively expected in generic conformal field theory. Nevertheless these results are consequences of $PSU(2,2|4)$ invariance.

- Instanton effects $\mathcal{N}=4$ SYM correspond to D-instanton effects in type IIB superstring. In particular, certain higher derivative terms in the superstring effective action on $AdS_5\times S^5$ are exactly reproduced by instanton dominated correlators on the boundary.

- (Partial) non-renormalization of "BPS" Wilson loops holds. For instance two parallel lines do not receive quantum corrections [36] while circular loops receive perturbative contributions only from rainbow diagrams [37] and non-perturbative contributions from instantons [38].

- The RG flows induced by deformations of the boundary CFT are holographically described by domain wall solutions in the bulk. In particular, it is possible to prove the holographic

c-theorem. Indeed one can build a holographic c-function [39]

$$\beta^i = \dot{\phi}^i = \frac{\phi'(r)}{A'(r)}\,, \qquad \dot{c}_H = -G_{ij}\beta^i\beta^j \leq 0\,, \tag{26}$$

and prove that it be monotonically decreasing along the flow.

- There is a nice way to reproduce anomaly in $\mathcal{N} = 4$ arising upon coupling the theory to external gravity or other backgrounds. In particular the holographic trace anomaly reads [40]
$$\langle \mathcal{T}^\mu_\mu \rangle_{(g_{\mu\nu})} = \frac{N^2}{4}(R_{\mu\nu}R^{\mu\nu} - \frac{1}{8}R^2)\,. \tag{27}$$
Quite remarkably, the structure of the anomaly implies
$$c_H = a_H \tag{28}$$
at least at large N. This is simple and powerful constraint on the CFT's that admit a holographic dual description. The techniques of holographic renormalization [41, 42, 43] have been developed to the point that one can reliably compute not only the spectrum of superglueball states [44] but also three-point amplitudes and the associated decay rates [45].

- Very encouraging results come from the recent work on string solitons with large spin or large charges [8], which qualitatively reproduce gauge theory expectations. In particular, the scaling
$$\sum_{k=1}^{S} \frac{1}{k} \sim \log S \tag{29}$$
for long strings with large spin S on $AdS_5 \times S^5$ has been found [8]. Moreover, the BMN limit [46], describing operators with large R charge, is believed to be dual to string theory on a pp-wave background emerging from the Penrose limit of $AdS_5 \times S^5$. For BMN operators, with a small number of impurities, light-cone quantization of the superstring suggests a close-form expression for the dimension as a function of coupling λ. For two-impurity BMN operators one has
$$\Delta = J + \sum_n N_n \sqrt{1 + \frac{\lambda n^2}{J^2}}\,. \tag{30}$$
where J is the R-charge.

4 Stringy $AdS_5 \times S^5$ and higher spin holography

In the boundary CFT the HS symmetry enhancement point is at $\lambda = 0$, so one may naively expect it to correspond to zero radius for $AdS_5 \times S^5$. Actually, there might be corrections to $R^2 = \alpha'\sqrt{\lambda}$ for $\lambda \ll 1$ and we would argue that it is not unreasonable to expect the higher-spin enhancement point to coincide with the self-dual point $R^2 \sim \alpha'$. Ideally, one would like to determine the string spectrum by (covariant) quantization in $AdS_5 \times S^5$ background. However the presence of a R-R background has prevented a satisfactory resolution of the problem so far despite some progress in this direction [47]. Since we are far from a full understanding of stringy effects at small radius we have to devise an alternative strategy.

In [1] we computed the KK spectrum by naive dimensional reduction on the sphere and then extrapolated it to small radius, *i.e.* to the HS symmetry enhancement point. As we momentarily

see, group theory techniques essentially determine all the quantum numbers, except for the scaling dimension, dual to the AdS mass. In order to produce a formula valid for all states at the HS point, we first exploit HS symmetry and derive a formula for the dimension of the massless HS fields. Then we take the BMN limit [46] as a hint and extend the formula so as to encompass the full spectrum. The final simple and effective formula does not only reproduce the HS massless multiplets as well as their KK excitations but does also describe genuinely massive states, which are always part of long multiplets. Finally, we compare the resulting string spectrum at the HS enhancement point in the bulk with the spectrum of free $\mathcal{N}=4$ SYM theory on the AdS boundary. Clearly the matching of the spectrum is a sign that we are on the right track, but it is by no means a rigorous proof.

In order to study the string spectrum on $AdS_5 \times S^5$, we started with the GS formalism and built the spectrum of type IIB superstrings in the light-cone gauge in flat ten-dimensional space-time. The little group is $SO(8)$ for massless states and $SO(9)$ for massive ones. The chiral worldsheet supermultiplets are described by

$$Q_s = 8_v - 8_s\,, \qquad Q_c = 8_v - 8_c\,, \tag{31}$$

(see [1] for notations and details). At level 0 one has the massless supergravity multiplet

$$l=0 \qquad Q_sQ_s = T_0 \qquad (\text{supergravity: } 128_B - 128_F) \tag{32}$$

At level 1 there are 2^{16} states as a result of the enhancement of $SO(8)$ to $SO(9)$ (128-fermions, 84-totally symmetric tensors and 44-antisymmetric)

$$l=1 \qquad Q_s^2Q_c^2 = T_1 \qquad (2^{16} \text{ states: } (44+84-128)^2) \tag{33}$$

Similarly at level l=2

$$l=2 \qquad Q_c^2(Q_s + Q_s\cdot Q_s) = T_1\times(1+8_v) = T_1\times(9) \tag{34}$$

and so on

$$l \qquad \cdots = T_1 \times v_l^2 \qquad (v_1 = 1, v_2 = 9, \dots)\,. \tag{35}$$

The important thing is that after building the spectrum one has to rewrite each level of the spectrum in terms of T_1, comprising 2^{16} states. Eventually, T_1 turns out to provide us with a representation of the superconformal group. What remains per each chirality will be called v_l. Combining with the opposite world-sheet chirality one gets v_l^2 as ground states.

In order to extend the analysis to $AdS_5 \times S^5$, *i.e.* perform the naive KK reduction, requires identifying which kinds of representations of the S^5 isometry group $SO(6)$ appear associated to a given representation of $SO(5)$. The latter arises from the decomposition $SO(9) \to SO(4) \times SO(5)$ for the massive states in flat space-time. Group theory yields the answer: only those representations of $SO(6)$ appear in the spectrum which contain the given representation of $SO(5)$ under $SO(6) \to SO(5)$.

Thus, after diagonalizing the wave equation for the bulk fields

$$\Phi(x,y) = \sum X_{AdS}(x)Y_{S^5}(y)$$

the spectrum of a string on $AdS_5 \times S^5$ assembles into representations of $SU(4) \approx SO(6)$, which are essentially given by spherical harmonics, with AdS mass

$$R^2M_\Phi^2 = \Delta(\Delta-4) - \Delta_{min}(\Delta_{min}-4) \leftrightarrow C_2[SU(2,2|4)]\,. \tag{36}$$

More explicitly, the wave equation can be written as

$$(\nabla^2_{AdS_5\times S^5} - M^2_\Phi)\Phi_{\{\mu\}\{i\}} = 0\,, \qquad \{\mu\}\in R_{SO(4,1)}\,, \quad \{i\}\in R_{SO(5)} \tag{37}$$

and one gets

$$\Phi_{\{\mu\}\{i\}} = \sum_{[kpq]} X^{[kpq]}_{\{\mu\}}(x) Y^{[kpq]}_{\{i\}}(y)\,, \tag{38}$$

where $[kpq]\in SO(6)$, the isometry group of S^5 and

$$\nabla^2_{S^5} Y^{[kpq]}_{\{i\}} = -\frac{1}{R^2}\Big(C_2[SO(6)] - C_2[SO(5)]\Big) Y^{[kpq]}_{\{i\}}\,. \tag{39}$$

The KK tower build on the top of $SO(5)$ representation is given the following direct sum of $SO(6)$ representation

$$KK_{[mn]} = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=m-r}^{\infty}[r+s;p;r+n-s] + \sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\sum_{p=m-r-1}^{\infty}[r+s+1;p;r+n-s]\,, \tag{40}$$

where $[mn]$ are the two Dynkin labels of an $SO(5)$-representation.

The remaining $SO(4,2)$ quantum numbers, required to perform a lift to HWS representations $\mathcal{D}$ of $PSU(2,2|4)\supset SO(4,2)\times SO(6)\supset SO(4)\times SO(5)$, are the Lorentz spins j_L and j_R, and the scaling dimension Δ

$$D\{\Delta;(j_L,j_R);[k,p,q]\} = (1+Q+\cdots+Q^{16})\Psi^{(\Delta,j_L,j_R)}_{[k,p,q]}\,. \tag{41}$$

For instance, at level $l=1$, which corresponds to $\mathcal{D}\{2,(00);[000]\}\equiv \hat{T}^{(2)}_1$, the spectrum of KK excitations assembles into an infinite number of $SU(2,2|4)$ representations

$$H^{KK}_1 = \sum_{M=0}^{\infty}[0n0]^M_{(0,0)}\hat{T}^{(2)}_1 = \sum_{M=0}^{\infty}\mathcal{D}\{\Delta_0 = 2+n,(00),[0n0]\}\,. \tag{42}$$

The HWS's in this formula have dimensions $\Delta_0 = 2+n$, spin-0 and belong to $SU(4)$ representation $[0n0]$, which describes exactly the spherical harmonics. As we have already mentioned, so far the dimension Δ_0, at the HS point, is postulated so as to get the correct massless HS fields in the bulk.

For $l=2$ the situation is slightly more involved, because **9** in (34) in v_l is neither a representation of the $SO(9,1)$ nor a representation of the $SO(4,2)\times SO(6)$. In this case, the correct way to proceed is to first decompose $\mathbf{9}\to\mathbf{10}-\mathbf{1}$ and then $\mathbf{10}\to\mathbf{6}+\mathbf{4}$

$$\underline{9}\to[010]^1_{(00)} + [000]^1_{(\frac12,\frac12)} - [000]^2_{(00)} \sim \underline{10}-\underline{1}\,. \tag{43}$$

The corresponding KK-tower has a form

$$H^{KK}_2 = \sum_{M=0}^{\infty}[0n0]^M_{00}\times\hat{T}^{(2)}_1\times\{[020]^2_{(00)} + [101]^2_{(00)} + [000]^2_{(00)} + 2[010]^2_{(\frac12,\frac12)} +$$

$$+[000]^2_{(11)} + [000]^2_{(10)} + [000]^2_{(01)} + [000]^2_{(00)} + [000]^4_{(00)} - 2[010]^3_{(00)} - 2[000]^3_{(\frac12,\frac12)}\}\,. \tag{44}$$

It is worth stressing that negative multiplicities cause no problem as they cancel in infinite sum over n, precisely when the dimension is chosen properly.

The analysis of higher levels is analogous though slightly more involved. One has

$$H_l = \sum_0^\infty [0n0]^M_{00} \times \hat{T}^{(2)}_l \times (v^2_l)\,, \tag{45}$$

with the decomposition

$$v^2_l = [000]^\Delta_{(l-1,l-1)} + \dots\,. \tag{46}$$

4.1 Exploiting HS symmetry

The superconformal group $PSU(2,2|4)$ admits a HS symmetry extension, called $HS(2,2|4)$ extension [48,12,13,14]. In [12,13,14] Sezgin and Sundell have shown that the superstring states belonging to the first Regge trajectory on AdS can be put in one to one correspondence with the physical states in the master fields of Vasiliev's theory [21]. The HS fields which have maximum spin, *i.e.* $S_{max} = 2l + 2$ at level l, are dual to twist 2 currents, which are conserved at vanishing coupling. Including their KK excitations, one is lead to conjecture the following formula for their scaling dimensions

$$\Delta_0 = 2l + n \tag{47}$$

at the HS enhancement point. Now, at $\lambda \neq 0$, as we said in the introduction, *"La Grande Bouffe"* happens, since HS multiplets start to "eat" lower spin multiplets. For example, the short and "massless" $\mathcal{N} = 4$ Konishi multiplet combines with three more multiplets

$$K_{\text{long}} = K_{\text{short}} + K_{\frac{1}{8}} + K^*_{\frac{1}{8}} + K_{\frac{1}{4}} \tag{48}$$

and becomes long and massive. The classically conserved currents in the Konishi multiplet are violated by the supersymmetric Konishi anomaly

$$\bar{D}^A_{\dot\alpha}\bar{D}^{B\dot\alpha}K = gTr(W^{AE}[W^{BF},\bar{W}_{EF}]) + g^2 D^\alpha_E D_{\alpha F} Tr(W^{AE}W^{BF})\,, \tag{49}$$

where W^{AB} is the twisted chiral multiplet describing the $\mathcal{N} = 4$ singleton. In passing, the anomalous dimension of the Konishi multiplet is known up to three loops [49,50,51]

$$\gamma_K^{1-\text{loop}} = 3\frac{g^2N}{4\pi^2}\,, \quad \gamma_k^{2-\text{loop}} = -3\frac{(g^2N)^2}{(4\pi^2)^2}\,, \quad \gamma_k^{3-\text{loop}} = 21\frac{(g^2N)^4}{(4\pi^2)^4} \quad , \tag{50}$$

whereas the anomalous dimensions of many other multiplets were computed by using both old-fashioned field-theoretical methods as well as modern and sophisticated techniques based on the integrability of the super-spin chain capturing the dynamics of the $\mathcal{N} = 4$ dilatation operator [51].

A systematic comparison with the operator spectrum of free $\mathcal{N} = 4$ SYM, may not forgo the knowledge of a mass formula encompassing all string states at the HS enhancement. Remarkably enough such a formula was "derived" in [2]. Consideration of the pp-wave limit of $AdS_5 \times S^5$ indeed suggests the following formula

$$\Delta = J + \nu\,, \tag{51}$$

where $\nu = \sum_n N_n$ and J is the R-charge emerging from $SO(10) \to SO(8)$ and N_n is the number of string excitations. Even though, the Penrose limit requires *inter alia* going to large radius so that the resulting BMN formula (51) is expected to be only valid for states with large R-charge J, (51) can be extrapolate to $\lambda \approx 0$ for all J's.

5 N=4 SYM spectrum: Polya(kov) Theory

In order to make a comparison of our previous results with the $\mathcal{N}=4$ SYM spectrum we have to devise strategy to enumerate SYM states, and the correct way to proceed is to use Polya theory [15]. The idea was first applied by A. Polyakov in [11] to the counting of gauge invariant operators made out only of bosonic "letters".

Let us start by briefly reviewing the basics of Polya theory. Consider a set of words $A, B, \ldots$ made out of n letters chosen within the alphabet $\{a_i\}$ with $i = 1, \ldots p$. Let G be a group action defining the equivalence relation $A \sim B$ for $A = gB$ with $g \in G \subset S_n$. Elements $g \in S_n$ can be divided into conjugacy classes $[g] = (1)^{b_1} \ldots (n)^{b_n}$, according to the numbers $\{b_k(g)\}$ of cycles of length k. Polya theorem states that the set of inequivalent words are generated by the formula:

$$P_G(\{a_i\}) \equiv \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{n} (a_1^k + a_2^k + \cdots + a_p^k)^{b_k(g)} . \quad (52)$$

In particular, for $G = Z_n$, the cyclic permutation subgroup of S_n, the elements $g \in G$ belong to one of the conjugacy classes $[g] = (d)^{\frac{n}{d}}$ for each divisor d of n. The number of elements in a given conjugacy class labelled by d is given by Euler's totient function $\varphi(d)$, that equals the number of integers relatively prime to and smaller than n. For $n = 1$ one defines $\varphi(1) = 1$. Computing P_G for $G = Z_n$ one finds:

$$P_n(\{a_i\}) \equiv \frac{1}{n} \sum_{d/n} \varphi(d)(a_1^d + a_2^d + \cdots + a_p^d)^{\frac{n}{d}} . \quad (53)$$

The number of inequivalent words can be read off from (52) by simply letting $a_i \to 1$.

For instance, the possible choices of "necklaces" with six "beads" of two different colors a and b, are given by

$$P_6(a, b) = \frac{1}{6}[(a + b)^6 + (a^2 + b^2)^3 + 2(a^3 + b^3)^3 + 2(a^6 + b^6)] =$$
$$= a^6 + a^5 b + 3a^4 b^2 + 4a^3 b^3 + 3a^2 b^4 + ab^5 + b^6 ,$$

and the number of different necklaces is $P_6(a = b = 1) = 14$.

We are now ready to implement this construction in $\mathcal{N}=4$ theory, where the letters are the fundamental fields together with their derivatives

$$\partial^s \varphi^i; \partial^s \lambda^A; \partial^s \bar{\lambda}_A; \partial^s F^+; \partial^s F^- .$$

There are the **6** scalar fields φ^i, **4** Weyl gaugini λ^A, $\mathbf{4}^*$ conjugate ones $\bar{\lambda}_A$ and the (anti) self-dual field strengths $F^\pm$. Since it is irrelevant for counting operators wether one is in free theory or not, we take as mass-shell conditions the free field equations. The single-letter on-shell partition functions then take the following form

$$Z_s(q) = \sum_\Delta n_\Delta^s q^\Delta = n_s \frac{q^{\Delta_s}}{(1-q)^4}(1 - q^2) , \qquad \Delta_s = 1 , \quad (54)$$

for the scalars such that $\partial^\mu \partial_\mu \varphi = 0$.

$$Z_f(q) = \sum_\Delta n_\Delta^f q^\Delta = n_f \frac{2q^{\Delta_f}}{(1-q)^4}(1 - q) , \qquad \Delta_f = 3/2 , \quad (55)$$

for the fermions with $\gamma^\mu \partial_\mu \lambda = 0$. For the vector field one gets a little bit involved expression, because apart from the equation of motion one has to take into account Bianchi identities $\partial^\mu F^+_{\mu\nu} = \partial^\mu F^-_{\mu\nu} = 0$

$$Z_v(q) = \sum_\Delta n^v_\Delta q^\Delta = n_v \frac{2q^{\Delta_f}}{(1-q)^4}(3 - 4q + q^2)\,, \qquad \Delta_f = 2\,. \tag{56}$$

Taking statistics into account for $U(N)$ we obtain the free SYM partition function

$$Z_{YM} = \sum_{M=1}^{N} \sum_{d|n} \frac{1}{n} \varphi(d) [Z_s(q^d) + +Z_v(q^d) - Z_f(q^d) - Z_{\bar{f}}(q^d)]^{n/d}\,. \tag{57}$$

In fact, (57) is not exactly a partition function, rather it is what one may call the Witten index, wherein fermions enter with a minus sign and bosons with a plus sign. Now, words that consist of more than N constituents decompose into multi-trace operators, so representing $n = kd$, where d is a divisor of n and summing over k and d independently in the limit $N \to \infty$ one gets

$$Z_{YM} = -\sum_d \frac{\varphi(d)}{d} \log[1 - Z_s(q^d) - Z_v(q^d) + Z_f(q^d) + Z_{\bar{f}}(q^d)]\,. \tag{58}$$

For $SU(N)$, we have to subtract words that consist of a single constituent, which are not gauge invariant, thus the sum starts with $n = 2$ or equivalently

$$Z^{SU(N)} = Z^{U(N)} - Z^{U(1)}\,. \tag{59}$$

Finally, for $\mathcal{N} = 4$ we have $n_s = 6, n_f = n_{\bar{f}} = 4, n_v = 1$. Plugging into (57) and expanding in powers of q up to $\Delta = 4$ yields

$$Z_{\mathcal{N}=4}(q) = 21q^2 - 96q^{5/2} + 361q^3 - 1328q^{7/2} + 4601q^4 + \ldots \tag{60}$$

5.1 Eratostene's (super)sieve

In order to simplify the comparison of the spectrum of SYM with the previously derived string spectrum, one can restrict the attention to superconformal primaries by means of Eratostenes's super-sieve, that allows us to get rid of the superdescendants. This procedure would be trivial if we knew that all multiplets were long, but unfortunately the partition function contains 1/2-BPS, 1/4-BPS, semishort ones too. The structure of these multiplets of $PSU(2,2|4)$ is more elaborated than the structure of long multiplets, which in turn is simply coded in and factorizes on the highest weight state.

Superconformal primaries, *i.e.* HWS of $SU(2,2|4)$, are defined by the condition

$$\hat{\delta}_S \mathcal{O} \equiv [\xi_A S^A + \bar{\xi}^A \bar{S}_A, \mathcal{O}] = 0\,, \tag{61}$$

where δ-is the supersymmetry transformation

$$\hat{\delta}_S = \delta_S - \delta_Q\,, \qquad (\eta = x - \xi)$$

$$\hat{\delta}_S \varphi^i = 0\,, \quad \hat{\delta}_S \lambda^A = \tau_i^{AB} \varphi^i \xi_B\,, \quad \hat{\delta}_S \bar{\lambda}_A = 0\,, \quad \hat{\delta}_S F_{\mu\nu} = \xi_A \sigma_{\mu\nu} \lambda^A$$

and τ_i^{AB} are 4×4 Weyl blocks of Dirac matrices in $d = 6$. The procedure can be implemented step by step using computer.

- Start with the lowest primaries – the Konishi scalar field $K_1 = tr\varphi_i\varphi^i$, and the lowest CPO $Q^{ij}_{20'} = tr\varphi^{(i}\varphi^{j)|}|)$
- Remove their superdescendants
- The first operator one finds at the lowest dimension is necessarily a superprimary.
- Go back to step 2

We have been able to perform this procedure up to $\Delta = 11.5$ and the agreement with "naive" superstring spectrum

$$H_l = \sum_{n,l}[0n0]_{(00)} \times \hat{H}_l^{flat}$$

is perfect! Let us stress once more that our mass formula (51), though derived exploing HS symmetry and suggested by the BMN formula extrapolated to the HS enhancement point, reproduces semishort as well as genuinely long multiplets. The latter correspond to massive string states which never get close to being massless.

6 HS extension of (P)SU(2,2|4)

In the second part of this lecture, we identify the HS content of $\mathcal{N} = 4$ SYM at the HS enhancement point. Since we focus on the higher spin extension of superconformal algebra, it is convenient to realize $SU(2,2|4)$ by means of (super)oscillators $\zeta_\Lambda = (y_a, \theta_A)$ with

$$[y_a, \bar{y}^b] = \delta_a{}^b\,, \qquad \{\theta_A, \bar{\theta}^B\} = \delta_A{}^B\,, \tag{62}$$

where $y_a, \bar{y}^b$ are bosonic oscillators with $a, b = 1, \ldots, 4$ and $\theta_A, \bar{\theta}^B$ are fermionic oscillators with $A, B = 1, \ldots, 4$. The $su(2,2|4)$ superalgebra is spanned by various traceless bilinears of these oscillators. There are generators,

$$J_a{}^b = \bar{y}^a y_b - \frac{1}{2}K\delta^a{}_b\,, \qquad K = \frac{1}{2}\bar{y}^a y_a \tag{63}$$

which represents $so(4,2) \oplus u(1)_K$ subalgebra and generators

$$T^A{}_B = \bar{\theta}^A\theta_B - \frac{1}{2}B\delta^A{}_B\,, \qquad B = \frac{1}{2}\bar{\theta}^A\theta_A \tag{64}$$

which correspond $su(4) \oplus u(1)_B$. The abelian charge B is to be identified with the generator of Intriligator's "bonus symmetry" dual to the anomalous $U(1)_B$ chiral symmetry of type IIB in the AdS bulk. The Poincaré and superconformal supercharges are of the form

$$Q^A_a = y_a\theta^A\,, \qquad \bar{Q}^a_A = \bar{y}^a\theta_A\,. \tag{65}$$

The combination

$$C = K + B = \frac{1}{2}\bar{\zeta}^\Lambda\zeta_\Lambda\,. \tag{66}$$

is a central charge that commutes with all the other generators. Since all of the fundamental fields $\{A_\mu, \lambda^\alpha_A, \bar{\lambda}^A_{\dot{\alpha}}, \varphi^i\}$ have central charge equal to zero, we expect that local composites, just as well, have central charge equal to zero. So we consistently put

$$C = 0\,. \tag{67}$$

The higher spin extension $hs(2,2|4)$ is generated by the odd powers of the above generators

$$hs(2,2|4) = Env(su(2,2|4))/I_C = \bigoplus_{l=0}^{\infty} A_{2l+1}\,, \tag{68}$$

where I_C is the ideal generated by C and the elements J_{2l+1} in A_{2l+1} are of the form

$$J_{2l+1} = P^{\Lambda_1\ldots\Lambda_{2l+1}}_{\Sigma_1\ldots\Sigma_{2l+1}}\bar{\zeta}^{\Sigma_1}\ldots\bar{\zeta}^{\Sigma_{2l+1}}\zeta_{\Lambda_1}\ldots\zeta_{\Lambda_{2l+1}} - \text{traces}\,. \tag{69}$$

The singleton representation of $su(2,2|4)$ turns out to be also the singleton of $hs(2,2|4)$ in such a way that any state in the singleton representation of $hs(2,2|4)$ can be reached from the HWS by one step using a single higher spin generator. Note, that in $su(2,2|4)$ the situation is different, namely in order to reach a generic descendant from the HWS one has to apply several times different rasing operators.

6.1 sl(2) and its HS extension hs(1,1)

Since the $hs(2,2|4)$ algebra is rather complicated, in order to clarify the algebraic construction, we make a short detour in what may be called the $hs(1,1)$ algebra, the higher spin extension of $sl(2)\approx su(1,1)$.

Consider the $sl(2)$ subalgebra:

$$[J_-, J_+] = 2J_3\,,\qquad [J_3, J_\pm] = \pm J_\pm\,. \tag{70}$$

This algebra can be represented in terms of oscillators

$$J_+ = a^+ + a^+a^+a\,,\qquad J_3 = \frac{1}{2} + a^+a\,,\qquad J_- = a\,, \tag{71}$$

where, as usual, $[a, a^+] = 1$ and the vacuum state $|0\rangle$ is annihilated by $J_- = a$. Other $sl(2)$ HWS's are defined by

$$J_- f(a^+)|0\rangle = 0 \quad\Rightarrow\quad f(a^+) = 1\,. \tag{72}$$

Any state $(a^+)^n|0\rangle$ in this defining representation can be generated from its HWS $|0\rangle$ by acting with J_+^n. Therefore $f(a^+)$ defines a single irreducible representation of $sl(2)$, which will be called *singleton* and denoted by V_F. The $sl(2)$ spin of V_F is $-J_3|0\rangle = -\frac{1}{2}|0\rangle$. The dynamics of this subsector is governed by a Heisenberg spin chain.

The embedding of $sl(2)$ in $\mathcal{N}=4$ SYM can be performed in different ways. In particular, the HWS can be identified with the scalar $Z=\varphi^5+i\varphi^6$ and its $sl(2)$ descendants can be generated by the action of the derivative along a chosen complex direction, for instance $D = D_1 + iD_2$,

$$(a^+)^n|0\rangle \quad\leftrightarrow\quad D^n Z\,. \tag{73}$$

The tensor product of L singletons may be represented in the space of functions $f(a^+_{(1)},\ldots,a^+_{(L)})$. The resulting representation is no longer irreducible. This can be seen by looking for $sl(2)$ HWS's

$$J_- f(a^+_{(1)},\ldots,a^+_{(L)}) = \sum_{s=1}^{L}\partial_s f(a^+_{(1)},\ldots,a^+_{(L)}) = 0\,. \tag{74}$$

There is indeed more than one solution to these equations given by all possible functions of the form $f_L(a^+_{(s)} - a^+_{(s')})$. The basis for $sl(2)$ HWS's can be taken to be

$$|j_1,\ldots,j_{L-1}\rangle = (a^+_{(L)} - a^+_{(1)})^{j_1}\ldots(a^+_{(L)} - a^+_{(L-1)})^{j_{L-1}}|0\rangle\,, \tag{75}$$

with spin $J_3 = \frac{1}{2} + \sum_s j_s$. In particular for $L = 2$ one finds the known result

$$V_F \times V_F = \sum_{j=0}^{\infty} V_j \,, \tag{76}$$

where V_j is generated by acting with J_+ on the HWS $|j\rangle = (a^+_{(2)} - a^+_{(1)})^j |0\rangle$.

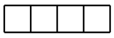
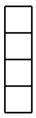
The higher spin algebra $hs(1,1)$ is generated by operators of the form

$$J_{p,q} = (a^+)^p a^q + \dots \,. \tag{77}$$

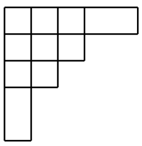
The generators $J_{p,q}$ with $p < q$ are raising operators. In the tensor product of L singletons, HWS's of $hs(1,1)$ are the solutions of

$$\sum_{i=1}^{L} (a^+_{(i)})^p \partial_i^q f(a^+_{(1)}, \dots, a^+_{(L)}) = 0 \,, \qquad p < q \,. \tag{78}$$

For $L = 2$ we can easily see that only two out of this infinite tower of HWS's survive for $j = 0$ and $j = 1$. That is all even objects belong to the same higher spin multiplet and all odd ones belong to another multiplet. For $L > 2$ one may consider either totally symmetric or totally antisymmetric representations. It can be easily shown that all of them are HWS's of HS multiplet.

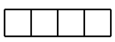
 $\Rightarrow |0\rangle_{(L)} \sim Z^L \,,$  $: \prod_{i<j} (a^+_i - a^+_j) |0\rangle_{(L)} \sim (ZDZ \dots D^{L-1} Z + a.s.) \,.$ (79)

For more complicated Young tableaux, where L boxes distributed in different k columns it can be shown, that there is a solution of the form

 $= \prod_{p=1}^{k} \prod_{i_p < j_p}^{L_p} (a^+_{i_p} - a^+_{j_p}) |0\rangle_{(L)} \leftrightarrow Z^{n_1} (DZ)^{n_2} \dots (D^{n_s} Z) + \text{perms} \,.$ (80)

The fact that (80) is indeed a solution is easily derived, however, its uniqueness is hard to prove.

The generalization to $hs(2,2|4)$ is almost straightforward for the totally symmetric representation

 HWS: $|0\rangle_{(L)} \leftrightarrow Z^L \,.$

Namely, one starts with 1-impurity states

$$(WZ^{L-1} + \text{symm.}) = \frac{1}{L} J^{HS}_{W\bar{Z}} Z^L \,, \tag{81}$$

where the impurity (W) appears symmetrically in all places, and proceeds with 2-impurity states

$$(W_1 Z^{k-2} W_2 Z^{L-k} + \text{symm.}) = \frac{1}{L(L-1)} J^{HS}_{W_1\bar{Z}} J^{HS}_{W_2\bar{Z}} Z^L \,, \tag{82}$$

and so on. Note, that all operators of this symmetry are descendants of Z^L due to the fact, that each state in a singleton representation can be reached by a single step starting from the highest weight state.

For generic Young tableaux the task is more involved. However, the above construction goes through and the same arguments hold. For example, besides the descendants $J_{W\bar{Z}}$ of Z^L there are $L-1$ 1-impurity multiplets of states associated to the $L-1$ Young tableaux with $L-1$ boxes in the first row and a single box in the second one. The vacuum state of HS multiplets associated to such tableaux can be taken to be $Y_{(k)} \equiv Z^k Y Z^{L-k-1} - Y Z^{L-1}$ with $k = 1, \ldots, L-1$. Any state with one impurity $Z^k W Z^{L-k-1} - W Z^{L-1}$ can be found by acting on $Y_{(k)}$ with the HS generators $J_{W\bar{Y}}$, where $J_{W\bar{Y}}$ is the HS generator that transforms Y into W and annihilates Z. The extension to other Young tableaux proceeds similarly though tediously.

6.2 HS content of $\mathcal{N} = 4$ SYM $\sim$ IIB at HS enhancement point

The free $\mathcal{N} = 4$ singleton partition function is given by the expression

$$Z_{\square} = \sum_{\Delta_s} (-1)^{2s} d_{\Delta_s} q^{\Delta_s} = \frac{2q(3+\sqrt{q})}{(1+\sqrt{q})^3}\,, \tag{83}$$

where Δ_s is the bare conformal dimension, *i.e.* the dimension at the HS enhancement point. Note that the singleton is not gauge invariant, thus one should build gauge invariant composites with two or more "letters". For example, the symmetric doubleton $\square\square$ can be obtained multiplying two singletons

$$\square \times \square = \square\square + \begin{array}{c}\square\\[-4pt]\square\end{array} \quad , \tag{84}$$

where the antisymmetric diagram appears only in interactions and must be neglected in the free theory. The spectrum of single-trace operators in $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group is given by all possible cyclic words built from letters chosen from $Z_{\square}$. It can be computed using Polya theory [15], which gives the generating function

$$Z(q, y_i) = \sum_{n>2} Z_n(q, y_i) = \sum_{n>2,d|n} u^n \frac{\varphi(d)}{n} Z_{\square}\,(q^d, y_i^d)^{\frac{n}{d}} \tag{85}$$

for cyclic words. The sum runs over all integers $n > 2$ and their divisors d, and $\varphi(d)$ is Euler's totient function, defined previously. The partition function (85) can be decomposed in representations of $hs(2,2|4)$, *i.e.* HS multiplets. In particular, all operators consisting of two letters, assemble into the (symmetric) doubleton.

$$Z_2 \equiv Z_{\square\square}^{\delta_{a,b}} = \sum_n \chi(v_{2n})\,. \tag{86}$$

For tri-pletons with three letters, one finds the totally symmetric tableau and the totally anti-symmetric one.

$$Z_3 = Z_{\square\square\square}^{(d_{abc})} + Z_{\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}}^{(f_{abc})} = \sum_{k=-\infty}^{+\infty} \sum_{n=0}^{\infty} c_n \Big(\chi(v_{2k}^n) + \chi(v_{2k+1}^{n+3}) + \chi(v_{2k+1}^n) + \chi(v_{2k}^{n+3}) \Big)\,, \tag{87}$$

where the coefficients $c_n \equiv 1 + [n/6] - \delta_{n,1}$, yielding the multiplicities of $psu(2,2|4)$ multiplets inside $hs(2,2|4)$, count the number of ways one can distribute derivatives (HS descendants) among the boxes of the tableaux. Similarly for the tetra-pletons and penta-pletons one finds

$$Z_4 = Z_{\square\square\square\square}^{(q_{abcd})} + Z_{\begin{array}{l}\square\square\\[-4pt]\square\\[-4pt]\square\end{array}}^{(d\cdot f)} + Z_{\begin{array}{l}\square\square\\[-4pt]\square\square\end{array}}^{(f\cdot f)} \tag{88}$$

$$Z_5 = Z^{(p_{abcde})}_{\ydiagram{5}} + Z_{\ydiagram{3,2}} + 2Z_{\ydiagram{3,1,1}} + Z_{\ydiagram{2,2,1}} + Z_{\ydiagram{1,1,1,1,1}} \tag{89}$$

In the above partition functions, totally symmetric tableaux are to be associated to KK descendants of the HS doubleton multiplet. Other tableaux are those associated to lower spin Stückelberg multiplets, that are needed in order for *"La Grande Bouffe"* to take place. We checked that multiplicities, quantum numbers and naive dimensions are correct so that they can pair with massless multiplets and give long multiplets. Finally there are genuinely massive representations that decompose into long multiplets of $su(2,2|4)$ even at the HS point.

7 Conclusions and outlook

Let us summarize the results presented in the lecture.

- There is perfect agreement between the string spectrum on $AdS_5 \times S^5$ "extrapolated" to the HS enhancement point with the spectrum of single trace gauge invariant operator in free $\mathcal{N} = 4$ SYM at large N.

- The massless doubleton comprises the HS gauge fields which are dual to the classically conserved HS currents. Massive YT-pletons, *i.e.* multiplets associated to Young tableau compatible with gauge invariance, correspond to KK excitations, Stückelberg fields and genuinely long and massive HS multiplet. *"La Grande Bouffe"* is kinematically allowed to take place at $\lambda \neq 0$.

- The one loop anomalous dimensions of the HS currents are given by

  $$\gamma_S^{1-\mathrm{loop}} = \sum_{k=1}^{S} \frac{1}{k} \, ,$$

  and it looks likely that it have a number theoretical origin.

- There are some interesting issues of integrability. First of all the dilatation operator can be identified with the Hamiltonian of a superspin chain and is integrable at one loop or in some sectors up to two and three loops. Flat currents in $AdS_5 \times S^5$ give rise to a Yangian structure. Finally, the HS gauge theory can be formulated as a Cartan integrable system.

- There are some surprising features in $\mathcal{N} = 4$ SYM that have emerged from resolving the operator mixing at finite N [52]. In particular there are "unprotected" operators with $\gamma_S^{1-\mathrm{loop}} = 0$ and there are operators with nonvanishing anomalous dimension, whose large N expansion truncates at some finite order in N [53]

  $$\gamma_s^{\text{1-loop}} = a + \frac{b}{N} + \frac{c}{N^2} \, ,$$

  with no higher order terms in $1/N$.

These and other facets of the AdS/CFT at small radius are worth further study in connection with integrability and HS symmetry enhancement. Sharpening the worldsheet description of the dynamics of type IIB superstrings on $AdS_5 \times S^5$ may turn to be crucial in all the above respects. Twelve dimensional aspects and two-time description [54] are worth exploring, too.

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Yang-Mills and N-homogeneous algebras

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Abstract. This is a review of some aspects of the theory of N-complexes and homogeneous algebras with applications to the cubic Yang-Mills algebra.

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Introduction

The aim of this lecture is to introduce some new algebraic techniques and apply them to the so-called Yang-Mills algebras in $(s+1)$-dimensional pseudo-Euclidean space. These algebras arise naturally from the Yang-Mills equations and are defined as the unital associative $\mathbb{C}$-algebras generated by the elements $\nabla_\lambda, \lambda \in \{0, \cdots, s\}$ with relations

$$g^{\lambda\mu}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0, \quad \forall \nu \in \{0, \dots, s\}.$$

Our algebraic tools will allow us to calculate the global dimension and the Hochschild dimension, among other homological properties, which will all be defined along the way.
The new algebraic techniques consist of an extension of the techniques used in the framework of quadratic algebras to the framework of N-homogeneous algebras with $N \geq 2$, ($N = 2$ corresponds to quadratic algebras). For a N-homogeneous algebra, the defining relations are homogeneous of degree N and it turns out that the natural generalisation of the Koszul complex of a quadratic algebra is here a N-complex.

In the first section we will introduce N-differentials and N-complexes and the corresponding generalisation of homology. Although this will be our main setting, we will not dwell long on that subject, as we are more interested in the properties of certain kinds of N-complexes, namely the Koszul N-complexes of N-homogeneous algebras. This will be the content of the second part of this lecture in which we will present all the tools necessary for our study of Yang-Mills algebras, which will be the central theme of the third part.

1 N-differentials and N-complexes

Let $\mathbb{K}$ be a commutative ring and E be a $\mathbb{K}$-module. We say that $d \in End(E)$ *is a N-differential* iff $d^N = 0$. Then (E, d) is a N-differential module. Since, by definition, $Im\ d^{N-p} \subset Ker\ d^p$, we have a generalization of homology

$$H_{(p)} = H_{(p)}(E) = \frac{Ker\, d^p}{Im\, d^{N-p}}, \quad \text{for } 1 \leq p < N.$$

$H_{(p)}(E)$ and $H_{(N-p)}(E)$ can also be obtained as the homology in degrees 0 and 1 of the $\mathbb{Z}_2$-complex

$$E \xrightarrow{d^p} E \xrightarrow{d^{N-p}} E.$$

For N-differential modules with $N \geq 3$, we have a basic lemma which has no analog for $N = 2$. Let $Z_{(n)} = Ker\ d^n$ and $B_{(n)} = Im\ d^{N-n}$. Since $Z_{(n)} \subset Z_{(n+1)}$ and $B_{(n)} \subset B_{(n+1)}$, the inclusion induces a morphism

$$[i] : H_{(n)} \to H_{(n+1)}.$$

On the other hand since $dZ_{(n+1)} \subset Z_{(n)}$ and $dB_{(n+1)} \subset B_{(n)}$, then d induces a morphism

$$[d] : H_{(n+1)} \to H_{(n)}.$$

One has the following lemma [DU-KE], [DU2].

Lemma 1.1. *Let ℓ and m be integers with $\ell \geq 1$, $m \geq 1$ and $\ell + m \leq N - 1$. Then the following*

hexagon $(\mathcal{H}^{\ell,m})$ *of homomorphisms*

$$\begin{tikzcd}
 & H_{(\ell+m)}(E) \arrow[r, "{[d]^m}"] & H_{(\ell)}(E) \arrow[dr, "{[i]^{N-(\ell+m)}}"] & \\
H_{(m)}(E) \arrow[ur, "{[i]^\ell}"] & & & H_{(N-m)}(E) \arrow[dl, "{[d]^\ell}"] \\
 & H_{(N-\ell)}(E) \arrow[ul, "{[d]^{N-(\ell+m)}}"] & H_{(N-(\ell+m))}(E) \arrow[l, "{[i]^m}"] &
\end{tikzcd}$$

is exact.

A N-complex of modules is a N-differential module E which is $\mathbb{Z}-$graded, $E = \bigoplus_{n\in\mathbb{Z}} E^n$, with N-differential d of degree 1 or -1. Then if we define

$$H^n_{(m)} = \frac{Ker(d^m : E^n \longrightarrow E^{n+m})}{d^{N-m}(E^{n+m-N})},$$

the modules $H_{(m)} = \bigoplus_{n\in\mathbb{Z}} H^n_{(m)}$ are $\mathbb{Z}$-graded modules. This is all we need to know about N-differentials and N-complexes.

Many examples of N-complexes are N-complexes associated with simplicial modules and root of the unit in a very general sense [DU1], [DU2], [DU-KE], [KA], [KA-WA], [MA], [WA] and it was shown in [DU2] that these N-complexes compute in fact the homology of the corresponding simplicial modules.
An interesting class of N-complexes which are not of the above type and which are relevant for higher-spin gauge theories is the class of N-complexes of tensor fields on $\mathbb{R}^n$ of mixed Young symmetry type defined in [DU-HE]. For these N-complexes which generalise the complex of differential forms on $\mathbb{R}^n$, a very nontrivial generalization of the Poincaré lemma was proved in [DU-HE].
The class of N-complexes of interest for the following also escape to the simplicial frame. For these N-complexes which generalise the Koszul complexes of quadratic algebras for algebras with relations homogeneous of degree N, $d^N = 0$ just reflect the fact that the relations are of degree N. From this point of view a natural generalization of the theory of N-homogeneous algebras and the associated N-complexes would be a theory of N-homogeneous operads with associated N-complexes in order to deal with structures where the product itself satisfies relations of degree N.

2 Homogeneous algebras of degree N

This part will be the longest and most demanding of this lecture and will, therefore, be subdivided into several subparts. We will start by defining the homogeneous algebras of degree N (or N-homogeneous algebras) and then proceed to several constructions involving these algebras which generalise what has already been done for quadratic algebras ($N = 2$). At the end of this first subpart we will introduce several complexes and N-complexes whose acyclicity is a too strong condition to cover interesting examples in case $N \geq 3$. This will be remedied in the next

subsection where we will introduce the Koszul complex whose acyclicity will give us a wealth of properties for the homogeneous algebra. Before reaping the benefits of Koszulity we will revise some concepts of homological algebra essential for the understanding of the definitions and properties that follow and which are the subject of the third and fourth subsections, and of the last section of this lecture as well.

2.1 The category $H_N Alg$

Let $\mathbb{K}$ be a field of characteristic 0, although for most of what follows this condition is not necessary, and let N be an integer with $N \geq 2$. Let E be a finite-dimensional vector space over $\mathbb{K}$ and $T(E)$ its tensor algebra ($T(E) = \mathbb{K} \bigoplus E \bigoplus E^{\otimes 2} \bigoplus \ldots = \bigoplus_{r \in \mathbb{N}} E^{\otimes r}$).

On what follows we will denote by E, E' and E'' (resp. R, R', R'') three arbitrary finite-dimensional vector spaces over $\mathbb{K}$ (resp. linear subspaces of $E^{\otimes N}, E'^{\otimes N}, E''^{\otimes N}$).

Definition 2.1. *A homogeneous algebra of degree N, or N-homogeneous algebra, is an algebra $\mathcal{A}$ of the form*

$$\mathcal{A} = A(E, R) = T(E)/(R).$$

where (R) is the two-sided ideal of $T(E)$ generated by a linear subspace R of $E^{\otimes N}$

Since R is homogeneous, $\mathcal{A}$ is a graded algebra, $\mathcal{A} = \oplus_{n \in \mathbb{N}} \mathcal{A}_n$ where

$$\mathcal{A}_n : \begin{cases} E^{\otimes n}, & n < N \\ E^{\otimes n} / \sum\limits_{r+s=n-N} E^r \otimes R \otimes E^s, & n \geq N. \end{cases}$$

Thus $\mathcal{A}$ is a graded algebra which is connected ($\mathcal{A}_0 = \mathbb{K}$) and generated in degree 1 ($\mathcal{A}_1 = E$).

Definition 2.2. *A morphism of N-homogeneous algebras $f : A(E, R) \to A(E', R')$ is a linear mapping $f : E \to E'$ such that $f^{\otimes N}(R) \subset R'$.*

Thus one has a *category $H_N Alg$ of N-homogeneous algebras* and the forgetful tensor

$$\begin{array}{lcl} H_N Alg & \to & Vect \\ \mathcal{A} = A(E, R) & \to & E \ , \end{array}$$

where Vect is the category of finite-dimensional vector spaces over $\mathbb{K}$.

Next, we shall look at some functors in $Vect$ and will try to lift them to $H_N Alg$. The first one is the *duality functor*. Given a N-homogeneous algebra $A(E, R)$ we define its dual $\mathcal{A}^!$ to be the N-homogeneous algebra $A^! = \mathcal{A}(E^*, R^\perp)$, where $R^\perp = \{\omega \in (E^{\otimes N})^* : \omega(x) = 0, \ \forall x \in R \subset (E^{\otimes N})^*\}$. Note that since E is finite-dimensional, we can identify $(E^{\otimes N})^*$ and $(E^*)^{\otimes N}$. One has canonically $(\mathcal{A}^!)^! = \mathcal{A}$ and if $f : \mathcal{A} \to \mathcal{B} = A(E', R')$ is a morphism in $H_N Alg$, then the transposed of $f : E \to E'$, is a linear mapping which induces the morphism $f^! : \mathcal{B}^! \to \mathcal{A}^!$. In other words, the duality functor is a contravariant involutive functor.

Now, we shall *lift the tensor product in the category Vect to the category $H_N Alg$*. The solution is not unique. We shall focus our attention to two liftings which can be considered as *maximal and minimal*, in the sense of the number of relations that are being imposed in the corresponding "tensor-product algebras". Moreover, these two tensor products will be the dual of each other in the category $H_N Alg$.

Let $\mathcal{A} = A(E, R)$ and $\mathcal{B} = A(E', R')$, and let Π_N be the permutation $(1, 2, \ldots, 2N) \to (1, N+1, 2, N+2, \ldots, k, N+k, \ldots, N, 2N)$ acting on the factors of the tensor products. We then define:

Definition 2.3. $\mathcal{A} \circ \mathcal{B} = A(E \otimes E', \Pi_N(R \otimes E'^{\otimes N} + E^{\otimes N} \otimes R'))$,

having the "biggest" relations and

Definition 2.4. $\mathcal{A} \bullet \mathcal{B} = A(E \otimes E', \Pi_N(R \otimes R'))$,

having the "smallest" relations.

From the identity $(R \otimes E'^{\otimes N} + E^{\otimes N} \otimes R')^{\perp} = R^{\perp} \otimes R'^{\perp}$, one has canonically

$$(\mathcal{A} \circ \mathcal{B})^! = \mathcal{A}^! \bullet \mathcal{B}^!$$

and

$$(\mathcal{A} \bullet \mathcal{B})^! = \mathcal{A}^! \circ \mathcal{B}^!.$$

From the inclusion $R \otimes R' \subset R \otimes E'^{\otimes N} + E^{\otimes N} \otimes R'$, one obtains the epimorphism $p : \mathcal{A} \bullet \mathcal{B} \to \mathcal{A} \circ \mathcal{B}$ in the category $H_N Alg$. In contrast to what happens in the quadratic case ($N = 2$), we have that for $N \geq 3$, $\mathcal{A} \otimes \mathcal{B}$ is not in general N-homogeneous. Nevertheless, we can still define an injective homomorphism

$$i : \mathcal{A} \circ \mathcal{B} \to \mathcal{A} \otimes \mathcal{B},$$

as follows. Let

$$\tilde{i} = \Pi_N^{-1} : (E \otimes E')^{\otimes n} \to E^{\otimes n} \otimes E'^{\otimes n}$$

which induces a monomorphism $\tilde{i} : T(E \otimes E') \to T(E) \otimes T(E')$. It is obvious that $\tilde{i}$ is an algebra homomorphism, and an isomorphism onto the subalgebra $\bigoplus_n E^{\otimes n} \otimes E'^{\otimes n}$ of $T(E) \otimes T(E')$. We have the following proposition [BE-DU-WA].

Proposition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two N-homogeneous algebras. Then $\tilde{i}$ passes to the quotient and induces an injective homomorphism i of algebras from $\mathcal{A} \circ \mathcal{B}$ to $\mathcal{A} \otimes \mathcal{B}$. The image of i is the subalgebra $\bigoplus_n (\mathcal{A}_n \otimes \mathcal{B}_n)$ of $\mathcal{A} \otimes \mathcal{B}$.*

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three N-homogeneous algebras corresponding to E, E' and E''. The isomorphisms $E \otimes E' \simeq E' \otimes E$ and $(E \otimes E') \otimes E'' \simeq E \otimes (E' \otimes E'')$ in the $Vect$ category induce the following isomorphisms in $H_N Alg$: $\mathcal{A} \circ \mathcal{B} \simeq \mathcal{B} \circ \mathcal{A}$ and $(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C} \simeq \mathcal{A} \circ (\mathcal{B} \circ \mathcal{C})$. Moreover the 1-dimensional vector space $\mathbb{K}t$ is a unit object in $(Vect, \otimes)$, $E \otimes \mathbb{K}t \simeq E$, which corresponds to the polynomial algebra $\mathbb{K}[t] = A(\mathbb{K}t, 0) = T(\mathbb{K})$ as a unit object of $(H_N Alg, \circ)$, that is $\mathcal{A} \circ \mathbb{K}[t] \simeq \mathbb{K}[t] \circ \mathcal{A} \simeq \mathcal{A}$ in $H_N Alg$.

This gives us part i) of the following proposition [BE-DU-WA] while part ii) which is the dual counterpart of part i) gives the very reason for the occurrence of N-complexes in this context.

Proposition 2.2. *i) $H_N Alg$ endowed with $\circ$ is a tensor category with unit object $\mathbb{K}[t]$.*
ii) $H_N Alg$ endowed with $\bullet$ is a tensor category with unit object $\Lambda_N\{d\} = \mathbb{K}[t]^!$.

The N-homogeneous algebra $\Lambda_N\{d\} = \mathbb{K}[t]^! \simeq T(\mathbb{K})/\mathbb{K}^{\otimes N}$ is the unital graded algebra generated in degree 1 by d with relation d^N=0, and a N-complex is just a graded $\Lambda_N(d)$-module.

2.2 The N-complexes $L(f)$ and $K(f)$

After all these preliminaries we are going to focus our attention to certain N-complexes. Let us recall the following result [BE-DU-WA].

Proposition 2.3. *The isomorphism*

$$Hom_{\mathbb{K}}(E \otimes E', E'') \simeq Hom_{\mathbb{K}}(E, E' \otimes E'')$$

in $Vect$ induces the isomorphism $Hom(\mathcal{A} \bullet \mathcal{B}, \mathcal{C}) \simeq Hom(\mathcal{A}, \mathcal{B}^! \circ \mathcal{C})$.

In this last proposition consider the case when $\mathcal{A} = \Lambda_N\{d\}$, and bearing in mind that $\Lambda_N\{d\}$ is a unit object in $H_N Alg$ endowed with $\bullet$, we obtain the following :

$$Hom(\mathcal{B}, \mathcal{C}) \simeq Hom(\Lambda_N\{d\}, \mathcal{B}^! \circ \mathcal{C}).$$

Let $\xi_f \in \mathcal{B}^! \circ \mathcal{C}$ be the image of d corresponding to $f \in Hom(\mathcal{B}, \mathcal{C})$. We have $\xi_f^N = 0$. We now use the injective algebra homomorphism

$$i : \mathcal{B}^! \circ \mathcal{C} \to \mathcal{B}^! \otimes \mathcal{C}.$$

and define d as the left multiplication by $i(\xi_f)$ in $\mathcal{B}^! \otimes \mathcal{C}$. One has $d^N = 0$. Hence, with the appropiate graduation $(\mathcal{B}^! \otimes \mathcal{C}, d)$ is a cochain N-complex of right $\mathcal{C}-$modules. Thus we define

Definition 2.5. *For any $f \in Hom(\mathcal{B}, \mathcal{C})$, $L(f) \stackrel{\text{def}}{=} (\mathcal{B}^! \otimes \mathcal{C}, d)$. When $\mathcal{B} = \mathcal{C}$ and $f = I_{\mathcal{B}}$ the N-complex $L(f)$ is denoted by $L(\mathcal{B})$.*

Actually what interests us is the dual of these complexes. Thus we define:

Definition 2.6. *For any $f \in Hom(\mathcal{B}, \mathcal{C})$, $K(f)$ is the chain N-complex of left $\mathcal{C}$-modules obtained by applying $Hom_{\mathcal{C}}(.,\mathcal{C})$ to each right $\mathcal{C}$-module of the N-complex $L(f) = (\mathcal{B}^! \otimes \mathcal{C}, d)$. As before when $\mathcal{B} = \mathcal{C}$ and $f = I_{\mathcal{B}}$ this complex $K(f)$ is denoted by $K(\mathcal{B})$.*

Now, as $Hom_{\mathcal{C}}(\mathcal{B}^!_n \otimes \mathcal{C}, \mathcal{C}) \simeq \mathcal{C} \otimes (\mathcal{B}^!_n)^*$, and the $(\mathcal{B}^!_n)$ are finite-dimensional vector spaces, we actually have that $K(f) = (\mathcal{C} \otimes \mathcal{B}^{!*}, d)$ where $d : \mathcal{C} \otimes (\mathcal{B}^!_{n+1})^* \to \mathcal{C} \otimes (\mathcal{B}^!_n)^*$.

We give now a more explicit description of $K(f)$ for any $f \in Hom(\mathcal{A}, \mathcal{B})$.

$$\mathcal{A}^!_n = E^{*\otimes n} \quad \text{for } n < N$$

and

$$\mathcal{A}^!_n = E^{*\otimes n} / \sum_{r+s=n-N} E^{*\otimes r} \otimes R^{\perp} \otimes E^{*\otimes s} \quad \text{for } n \geq N.$$

From this we obtain the following description of the dual spaces:

$$(\mathcal{A}^!_n)^* \simeq E^{\otimes n} \quad \text{if } n < N$$

and

$$(\mathcal{A}^!_n)^* \simeq \bigcap_{r+s=n-N} E^{\otimes r} \otimes R \otimes E^{\otimes s} \quad \text{if } n \geq N$$

which in either case imply $(\mathcal{A}^!_n)^* \subset E^{\otimes n}$. The N-differential d of $K(f)$ is induced by the linear mappings

$$c \otimes (e_1 \otimes \ldots \otimes e_n) \mapsto cf(e_1) \otimes (e_2 \otimes \ldots \otimes e_n)$$

of $\mathcal{B} \otimes E^{\otimes n}$ into $\mathcal{B} \otimes E^{\otimes (n-1)}$. One has $d(\mathcal{B}_s \otimes (\mathcal{A}^!_r)^*) \subset \mathcal{B}_{s+1} \otimes (\mathcal{A}^!_{r-1})^*)$, so that the N-complex $K(f)$ splits into sub-N-complexes $K(f)^n = \oplus_m \mathcal{B}_{n-m} \otimes (\mathcal{A}^!_m)^*$ with $n \in \mathbb{N}$. For instance $K(f)^0 = \mathcal{B}_0 \otimes (\mathcal{A}^!_0)^* = \mathbb{K} \otimes \mathbb{K} \simeq \mathbb{K}$ which gives

$$\cdots \to 0 \to \mathbb{K} \to 0 \to \cdots$$

For $1 \leq n \leq N-1$, $K(f)^n = \oplus_m E'^{\otimes n-m} \otimes E^{\otimes m}$ and therefore

$$0 \to E^{\otimes n} \stackrel{f\otimes I_E^{\otimes(n-1)}}{\longrightarrow} E' \otimes E^{\otimes(n-1)} \longrightarrow \cdots \stackrel{I_E'^{\otimes(n-1)}\otimes f}{\longrightarrow} E'^{\otimes n} \to 0$$

while $K(f)^N$ reads

$$0 \to R \stackrel{f\otimes I_E^{\otimes(N-1)}}{\longrightarrow} E' \otimes E^{\otimes(N-1)} \longrightarrow \cdots \longrightarrow E'^{\otimes N-1} \otimes E \stackrel{can}{\longrightarrow} \mathcal{B}_N \longrightarrow 0,$$

where *can* is the composition of $I_{E'}^{\otimes(N-1)} \otimes f$ with the canonical projection of $E'^{\otimes N}$ onto $E'^{\otimes N}/R' = \mathcal{B}_N$.

Next we will seek conditions for the *maximal acyclicity of* $K(f)$. For $n \leq N-2$, $K(f)^n$ is acyclic iff $E = E' = 0$. We then have the following result [BE-DU-WA] .

Proposition 2.4. *The N-complexes $K(f)^{N-1}$ and $K(f)^N$ are acyclic iff f is an isomorphism of N-homogeneous algebras.*

If we assume that $K(f)^{N-1}$ and $K(f)^N$ are acyclic we can identify the two N-homogeneous algebras $A(E,R)$ and $A(E',R')$ and take $f = I_{\mathcal{A}}$, so that we are dealing with $K(f) = K(\mathcal{A})$. Here the key result is the following [BE-DU-WA]:

Proposition 2.5. *For $N \geq 3$, $K(\mathcal{A})^n$ is acyclic $\forall n \geq N-1$ iff $R = 0$ or $R = E^{\otimes N}$.*

When $R = 0$, then $\mathcal{A} = T(E)$, and when $R = E^{\otimes N}$, then $\mathcal{A} = T(E^*)^!$ so for $N \geq 3$ the acyclicity of $K(\mathcal{A})$ for $n \geq N-1$ does not lead to an interesting class of algebras. Thus we have to look elsewhere for a generalization of Koszulity when $N \geq 3$.

2.3 Koszul homogeneous algebras.

Let us turn again to the N-complex $K(\mathcal{A})$

$$\cdots \longrightarrow \mathcal{A} \otimes (\mathcal{A}_i^!)^* \stackrel{d}{\longrightarrow} \mathcal{A} \otimes (\mathcal{A}_{i-1}^!)^* \longrightarrow \cdots \longrightarrow \mathcal{A} \otimes (\mathcal{A}_1^!)^* \stackrel{d}{\longrightarrow} \mathcal{A} \to 0.$$

We can contract this complex in many ways to obtain a 2-complex, denoted by $C_{p,r}$

$$\cdots \stackrel{d^{N-p}}{\longrightarrow} \mathcal{A} \otimes (\mathcal{A}_{N+r}^!)^* \stackrel{d^p}{\longrightarrow} \mathcal{A} \otimes (\mathcal{A}_{N-p+r}^!)^* \stackrel{d^{N-p}}{\longrightarrow} \mathcal{A} \otimes (\mathcal{A}_r^!)^* \stackrel{d}{\longrightarrow} \mathcal{A} \longrightarrow 0$$

where $0 \leq r \leq N-2$ and $r+1 \leq p \leq N-1$. One has the following result [BE-DU-WA]:

Proposition 2.6. *Let $\mathcal{A} = A(E,R)$ be a homogeneous algebra of degree $N \geq 3$. Assume that $(p,r) \neq (N-1,0)$ and that $C_{p,r}$ is exact at degree $i = 1$, then $R = 0$ or $R = E^{\otimes N}$.*

From this we conclude that acyclicity for the complex $C_{p,r}$, $(p,r) \neq (N-1,0)$, does not lead to any interesting condition. On the other hand, this result seems to single out the complex $C_{N-1,0}$.

Definition 2.7. *We will call $C_{N-1,0}$ the Koszul complex of $\mathcal{A}$ and $\mathcal{A}$ will be said to be Koszul when this complex is acyclic at degree $i > 0$. $C_{N-1,0}$ will be denoted $\mathcal{K}(\mathcal{A},\mathbb{K})$ as it coincides with the Koszul complex introduced by Roland Berger [BE1].*

Before going on, it will be necessary to review some concepts of homological algebra.

2.4 The global dimension of $\mathcal{A}$

Let A be a ring and P an A-module. P is said to be *projective* iff $Hom(P,.)$ transforms exact sequences into exact sequences, i.e., if

$$0 \longrightarrow B' \xrightarrow{f} B \xrightarrow{g} B'' \longrightarrow 0$$

is an exact sequence of A-modules then

$$0 \longrightarrow Hom(P,B') \xrightarrow{f_*} Hom(P,B) \xrightarrow{g_*} Hom(P,B'') \longrightarrow 0$$

also is. In particular free modules are projective. One can immediately see that this is equivalent to the following property. Given g an epimorphism, there is a morphism $\gamma : P \longrightarrow B$ that closes the following diagram,

$$\begin{array}{ccc} & P & \\ & \downarrow \alpha & \\ B \xrightarrow{g} & B'' & \rightarrow 0 \end{array}$$

i.e., $\alpha = g\gamma$. This fact allows us to construct for every A-module B a projective resolution, that is, for every A-module B one can construct an exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0$$

in which each P_i is projective.

Definition 2.8. *For an A-module B, we say that B has* projective dimension *$pd(B) \leq n$, if there is a projective resolution*

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0.$$

If no such resolution exists, we define $pd(B) = \infty$, and if n is the least such integer, then $pd(B) = n$.

Notice that $pd(B) = 0$ iff B is projective, so that the projective dimension is a way to measure how far a module is from being projective.

Definition 2.9. *For the ring $\mathcal{A}$ we define its* global dimension *as*

$$gldim(\mathcal{A}) = \sup\{pd(B) \,:\, B \textit{ is an } \mathcal{A}\textit{-module}\}.$$

We now turn back to the case where $\mathcal{A}$ is a N-homogeneous algebra. At first sight it may seem difficult to compute $gldim(\mathcal{A})$. Fortunately the following property comes to our rescue. One can regard the field $\mathbb{K}$ as an $\mathcal{A}$-module via the projection $\mathcal{A} \xrightarrow{\epsilon} \mathbb{K}$ for which $Ker\epsilon = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$, and defining $a.l = \epsilon(a)l$ for $a \in \mathcal{A}$ and $l \in \mathbb{K}$. It turns out that [RO]

Proposition 2.7.

$$gldim(\mathcal{A}) = pd_{\mathcal{A}}(\mathbb{K}).$$

so that to compute the global dimension of $\mathcal{A}$ we only have to find a projective resolution of $\mathbb{K}$ as an A-module which is minimal. In our case, to obtain a minimal projective resolution of $\mathbb{K}$ (one in which all the modules will actually be free) we turn to the complex $C_{N-1,0} = \mathcal{K}(\mathcal{A},\mathbb{K})$,

$$\cdots \xrightarrow{d} \mathcal{A} \otimes (\mathcal{A}^!_N)^* \xrightarrow{d^{N-1}} \mathcal{A} \otimes (\mathcal{A}^!_1)^* \xrightarrow{d} \mathcal{A} \longrightarrow 0. \quad (1)$$

Observe that the last d is not an epimorphism. By its definition $Im\, d = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$, so that we can complete $C_{N-1,0}$ on the right by

$$\mathcal{A} \otimes (\mathcal{A}_1^!)^* \xrightarrow{d} \mathcal{A} \xrightarrow{\epsilon} \mathbb{K} \to 0, \tag{2}$$

since $Im\, d = Ker\, \epsilon$, and ϵ is a projection this is exact. If $\mathcal{A}$ is Koszul then the whole sequence is exact and gives us a projective resolution of $\mathbb{K}$. Moreover it can be proved that this resolution is minimal. We conclude then that [BE-MA]

Proposition 2.8.

$$gldim(\mathcal{A}) = pd_{\mathcal{A}}(\mathbb{K}) = \max\{i : \mathcal{A}^!_{n(i)} \neq 0\}$$

with $n(2p) = Np$ *and* $n(2p+1) = Np + 1$.

2.5 Bicomplexes and the Gorenstein property

We have already defined the N-complex of left $\mathcal{A}$-modules

$$\cdots \xrightarrow{d} \mathcal{A} \otimes (\mathcal{A}^!_{n+1})^* \xrightarrow{d} \mathcal{A} \otimes (\mathcal{A}^!_n)^* \xrightarrow{d} \cdots \tag{3}$$

where d is induced by $a \otimes (e_1 \otimes \ldots \otimes e_n) \mapsto ae_1 \otimes (e_2 \otimes \ldots \otimes e_n)$. In the same way we can define the N-complex $\tilde{K}(\mathcal{A})$ of right $\mathcal{A}$-modules

$$\cdots \xrightarrow{\tilde{d}} (\mathcal{A}^!_{n+1})^* \otimes \mathcal{A} \xrightarrow{\tilde{d}} (\mathcal{A}^!_n)^* \otimes \mathcal{A} \xrightarrow{d} \cdots$$

where this time $\tilde{d}$ is induced by

$$(e_1 \otimes \ldots \otimes e_{n+1}) \otimes a \mapsto (e_1 \otimes \ldots \otimes e_n) \otimes e_{n+1}a.$$

All this allows us to construct two complexes of bimodules (L, R)

$$\cdots \xrightarrow{d_L, d_R} \mathcal{A} \otimes (\mathcal{A}^!_{n+1})^* \otimes \mathcal{A} \xrightarrow{d_L, d_R} \mathcal{A} \otimes (\mathcal{A}^!_n)^* \otimes \mathcal{A} \xrightarrow{d_L, d_R} \cdots$$

where $d_L = d \otimes I_{\mathcal{A}}$ and $d_R = I_{\mathcal{A}} \otimes \tilde{d}$. Since $d_L d_R = d_R d_L$,

$$(d_L - d_R)\,(\sum_{p=0}^{N-1} d_L^p d_R^{N-p-1}) = (\sum_{p=0}^{N-1} d_L^p d_R^{N-p-1})(d_L - d_R) = d_L^N - d_R^N = 0. \tag{4}$$

Finally we can define a chain complex $\mathcal{K}(\mathcal{A}, \mathcal{A})$ of $(\mathcal{A}, \mathcal{A})$-bimodules by

$$\mathcal{K}_{2m}(\mathcal{A}, \mathcal{A}) = \mathcal{A} \otimes (\mathcal{A}^!_{Nm})^* \otimes \mathcal{A} = \mathcal{K}_{2m}(\mathcal{A}, \mathbb{K}) \otimes \mathcal{A},$$

$$\mathcal{K}_{2m+1}(\mathcal{A}, \mathcal{A}) = \mathcal{A} \otimes (\mathcal{A}^!_{Nm+1})^* \otimes \mathcal{A} = \mathcal{K}_{2m+1}(\mathcal{A}, \mathbb{K}) \otimes \mathcal{A},$$

with differential δ' defined by

$$\delta' = d_L - d_R : \mathcal{K}_{2m+1}(\mathcal{A}, \mathcal{A}) \to \mathcal{K}_{2m}(\mathcal{A}, \mathcal{A}),$$

$$\delta' = \sum_{p=0}^{N-1} d_L^p d_R^{N-p-1} : \mathcal{K}_{2m}(\mathcal{A}, \mathcal{A}) \to \mathcal{K}_{2m-1}(\mathcal{A}, \mathcal{A}).$$

One can prove the following proposition, [BE1] with correction [BE2]

Proposition 2.9. *Let $\mathcal{A} = A(E,R)$ be a homogeneous algebra of degree N. One has $H_n\mathcal{K}(\mathcal{A},\mathcal{A}) = 0$ for $n \geq 1$ iff $\mathcal{A}$ is Koszul.*

We now define $\mathcal{A}^{op}$ the opposite algebra of $\mathcal{A}$ where multiplication is the opposite to that in $\mathcal{A}$, that is,

$$\text{in } \mathcal{A}^{op}, \quad r.s = sr.$$

Let $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$ and M be an $(\mathcal{A},\mathcal{A})$-bimodule. Then M can be regarded as a left $\mathcal{A}^e$-module M_l via the action $(a \otimes b)m = amb$, or as a right $\mathcal{A}^e$-module M_r via the action $m(a \otimes b) = bma$. Since $(\mathcal{A}^e)^{op}$ is isomorphic to $\mathcal{A}^e$ then $pd_{\mathcal{A}^e}M_l = pd_{\mathcal{A}^e}M_r$, which is denoted as $pd_{\mathcal{A}^e}M$.

Definition 2.10. *The number $pd_{\mathcal{A}^e}\mathcal{A}$ is called the* Hochschild dimension of *$\mathcal{A}$.*

Exactly as before, $\mathcal{A}$ being Koszul, in order to calculate the Hochschild dimension of $\mathcal{A}$ it will be enough to turn to the resolution $\mathcal{K}(\mathcal{A},\mathcal{A}) \to \mathcal{A} \to 0$ which is also minimal. For this reason, in our case the Hochschild dimension of $\mathcal{A}$ is equal to its global dimension, that is, we have

Proposition 2.10.

$$pd_{\mathcal{A}^e}\mathcal{A} = gldim\mathcal{A}.$$

Suppose now that $\mathcal{A}$ is Koszul and has finite global dimension D; we have seen in (1), and (2) that $\mathbb{K}$ admits a projective resolution which for simplicity we write as

$$0 \longrightarrow B_D \longrightarrow \cdots \longrightarrow B_1 \longrightarrow \mathcal{A} \xrightarrow{\epsilon} \mathbb{K} \longrightarrow 0.$$

We can construct the following 2-complex $\mathcal{C}$

$$0 \to Hom(\mathcal{A},\mathcal{A}) \to Hom(B_1,\mathcal{A}) \to \cdots \to Hom(B_D,\mathcal{A}) \to 0$$

which in general will not be exact, and so we can calculate its homology. Let $Ext^i_{\mathcal{A}}(\mathbb{K},\mathcal{A}) = H^i(\mathcal{C})$.

Definition 2.11. *We say that $\mathcal{A}$ is* AS-Gorenstein *if $Ext^i_{\mathcal{A}}(\mathbb{K},\mathcal{A}) = 0$ for $i \neq D$ and $Ext^D_{\mathcal{A}}(\mathbb{K},\mathcal{A}) = \mathbb{K}$.*

This property implies a precise form of Poincaré-like duality between the Hochschild homology and the Hochschild cohomology of $\mathcal{A}$ [BE-MA].

We have now in our hands all the necessary results to study Yang-Mills algebras. This will be the subject of the next section.

3 Yang-Mills algebras

Let $\mathcal{A}$ be the unital associative $\mathbb{C}$-algebra generated by the elements $\nabla_\lambda, \lambda \in \{0,\cdots,s\}$, with relations

$$g^{\lambda\mu}[\nabla_\lambda,[\nabla_\mu,\nabla_\nu]] = 0, \quad \forall \nu \in \{0,\ldots,s\}.$$

Giving degree 1 to the ∇_λ, $\mathcal{A}$ is a graded connected ($\mathcal{A}_0 = \mathbb{C}1$) algebra generated in degree one. In fact $\mathcal{A}$ is a 3-homogeneous algebra $\mathcal{A} = \mathcal{A}(E,R)$, where $E = \oplus_\lambda \mathbb{C}\nabla_\lambda$ and $R \subset E^{\otimes 3}$ is spanned by the

$$r_\nu = g^{\lambda\mu}(\nabla_\lambda \otimes \nabla_\mu \otimes \nabla_\nu + \nabla_\nu \otimes \nabla_\lambda \otimes \nabla_\mu - 2\nabla_\lambda \otimes \nabla_\nu \otimes \nabla_\mu) \tag{5}$$

for $\nu \in \{0,\ldots,s\}$. The cubic algebra $\mathcal{A}$ will be refered to as the *cubic Yang-Mills algebra*. The dual algebra $\mathcal{A}^!$ of $\mathcal{A}$ is the 3-homogeneous algebra $\mathcal{A}^! = \mathcal{A}(E^*,R^\perp)$, where $E^* = \oplus_\lambda \mathbb{C}\theta^\lambda$ is the

dual vector space of E with $\theta^\lambda(\nabla_\mu)=\delta^\lambda_\mu$ and where $R^\perp\subset E^{*\otimes 3}$ is the annihilator of R. One can verify that $\mathcal{A}^!$ is the unital associative $\mathbb{C}$-algebra generated by the θ^λ, $\lambda\in\{0,1,\dots,s\}$ with relations

$$\theta^\lambda\theta^\mu\theta^\nu=\frac{1}{s}(g^{\lambda\mu}\theta^\nu+g^{\mu\nu}\theta^\lambda-2g^{\lambda\nu}\theta^\mu)g,\ \text{where } g=g_{\alpha\beta}\theta^\alpha\theta^\beta\in\mathcal{A}^!_2. \tag{6}$$

Contracting (6) with $g_{\lambda\mu}$ one obtains $g\theta^\nu=\theta^\nu g$, so that g is central.

Proposition 3.1. *One has* $\mathcal{A}^!_0=\mathbb{C}1$, $\mathcal{A}^!_1=\oplus_\lambda\mathbb{C}\theta^\lambda$, $\mathcal{A}^!_2=\oplus_{\mu\nu}\mathbb{C}\theta^\mu\theta\nu$, $\mathcal{A}^!_3=\oplus_\lambda\mathbb{C}\theta^\lambda g$, $\mathcal{A}^!_4=\mathbb{C}g^2$ *and* $\mathcal{A}^!_n=0$, *for* $n\geq 5$.

The first three equalities follow from the definition, and the last three are established using (6). One has then the following dimensions:

$$dim(\mathcal{A}^!_0)=dim(\mathcal{A}^!_4)=1$$

$$dim(\mathcal{A}^!_1)=dim(\mathcal{A}^!_3)=s+1$$

$$dim(\mathcal{A}^!_2)=(s+1)^2 \text{ and } dim(\mathcal{A}^!_n)=0 \text{ otherwise.}$$

We now state our main result [CO-DU]:

Theorem 3.2. *The Yang-Mills algebra $\mathcal{A}$ is Koszul of global dimension 3 and Gorenstein.*

Proof. As has already been explained, to $\mathcal{A}$ is associated the chain 3-complex of left $\mathcal{A}$-modules $K(\mathcal{A})$,

$$0\to\mathcal{A}\otimes(\mathcal{A}^!_4)^*\xrightarrow{d}\mathcal{A}\otimes(\mathcal{A}^!_3)^*\xrightarrow{d}\mathcal{A}\otimes(\mathcal{A}^!_2)^*\xrightarrow{d}\mathcal{A}\otimes(\mathcal{A}^!_1)^*\xrightarrow{d}\mathcal{A}\to 0$$

where in this case

$$(\mathcal{A}^!_1)^*=E,\ (\mathcal{A}^!_2)^*=E^{\otimes 2},\ (\mathcal{A}^!_3)^*=R,\ (\mathcal{A}^!_4)^*=(E\otimes R)\bigcap(R\otimes E)$$

and d as in (3). From $K(\mathcal{A})$ one extracts the complex of left $\mathcal{A}$-modules $C_{2,0}=\mathcal{K}(\mathcal{A},\mathbb{C})$

$$0\to\mathcal{A}\otimes(\mathcal{A}^!_4)^*\xrightarrow{d}\mathcal{A}\otimes(\mathcal{A}^!_3)^*\xrightarrow{d^2}\mathcal{A}\otimes(\mathcal{A}^!_1)^*\xrightarrow{d}\mathcal{A}\to 0.$$

We will now show that this complex is acyclic. By the above Proposition 3.1 on the duals we know that $\mathcal{A}\otimes(\mathcal{A}^!_4)^*\simeq\mathcal{A}$ and $\mathcal{A}\otimes(\mathcal{A}^!_3)^*\simeq\mathcal{A}\otimes(\mathcal{A}^!_1)^*\simeq\mathcal{A}^{s+1}$.

On $\mathcal{A}\otimes(\mathcal{A}^!_4)^*$, d will work as follows; let α belong to $\mathcal{A}\otimes(\mathcal{A}^!_4)^*$. It can easily be checked that the element $g^{\mu\nu}\nabla_\mu\otimes r_\nu$ belongs to $(\mathcal{A}^!_4)^*$, so that it generates $(\mathcal{A}^!_4)^*$ which by Proposition 3.1 has dimension 1. Hence α can be written as $\alpha=a\otimes(g^{\mu\nu}\nabla_\mu\otimes r_\nu)$. Then $d\alpha=a\nabla_\mu\otimes r^\mu$, so that d can be viewed as

$$\begin{array}{rccc}\nabla^t: & \mathcal{A} & \to & \mathcal{A}^{s+1}\\ & a & \to & (a\nabla_0,\dots,a\nabla_s).\end{array}$$

On $\mathcal{A}\otimes(\mathcal{A}^!_3)^*$ d^2 works as follows; let

$$\alpha=a^\nu\otimes r_\nu=a_\mu\otimes g^{\mu\nu}g^{\alpha\beta}(\nabla_\alpha\otimes\nabla_\beta\otimes\nabla_\nu+\nabla_\nu\otimes\nabla_\alpha\otimes\nabla_\beta-2\nabla_\alpha\otimes\nabla_\nu\otimes\nabla_\beta),$$

$$\begin{aligned}d^2\alpha&=a_\mu g^{\mu\nu}g^{\alpha\beta}(\nabla_\alpha\nabla_\beta\otimes\nabla_\nu+\nabla_\nu\nabla_\alpha\otimes\nabla_\beta-2\nabla_\alpha\nabla_\nu\otimes\nabla_\beta)\\&=a_\mu(g^{\mu\nu}g^{\alpha\beta}+g^{\mu\alpha}g^{\nu\beta}-2g^{\mu\beta}g^{\nu\alpha})\nabla_\alpha\nabla_\beta\otimes\nabla_\nu=a_\mu M^{\mu\nu}\otimes\nabla_\nu\end{aligned}$$

where

$$M^{\mu\nu}=(g^{\mu\nu}g^{\alpha\beta}+g^{\mu\alpha}g^{\nu\beta}-2g^{\mu\beta}g^{\nu\alpha})\nabla_\alpha\nabla_\beta.$$

And d on $\mathcal{A}\otimes(\mathcal{A}^!_1)^*$ works as follows; let $\alpha = a^\nu \otimes \nabla_\nu$ $d\alpha = a^\nu\nabla_\nu$ so that d can be viewed as $\nabla : \mathcal{A}^{s+1} \to \mathcal{A}$,

$$(a_0,\dots,a_s) \mapsto (a_0,\dots,a_s)\begin{pmatrix}\nabla_0\\ .\\ .\\ .\\ \nabla_s\end{pmatrix}.$$

Then the complex $\mathcal{K}(\mathcal{A},\mathbb{C})$ can be written as:

$$0 \to \mathcal{A} \xrightarrow{\nabla^t} \mathcal{A}^{s+1} \xrightarrow{M} \mathcal{A}^{s+1} \xrightarrow{\nabla} \mathcal{A} \to 0$$

with $M\nabla^t = 0$ and $\nabla M = 0$ by (5). The exactitude of $\mathcal{K}(\mathcal{A},\mathbb{C})$ in degrees ≥ 1 follows easily from the definitions. We conclude that

$$0 \xrightarrow{i} \mathcal{A} \xrightarrow{\nabla^t} \mathcal{A}^{s+1} \xrightarrow{M} \mathcal{A}^{s+1} \xrightarrow{\nabla} \mathcal{A} \xrightarrow{\epsilon} \mathbb{C} \to 0 \tag{7}$$

is a minimal projective resolution of $\mathbb{C}$ viewed as an $\mathcal{A}$-module. Hence by Proposition 2.10 we have that $gldim\mathcal{A} = 3$.

If we now apply $Hom(.,\mathcal{A})$ to the resolution (7), we obtain

$$0 \to Hom(\mathbb{C},\mathcal{A}) \xrightarrow{\epsilon^*} Hom(\mathcal{A},\mathcal{A}) \xrightarrow{\nabla^*} Hom(\mathcal{A}^{s+1},\mathcal{A}) \xrightarrow{M^*} Hom(\mathcal{A}^{s+1},\mathcal{A}) \xrightarrow{\nabla^{t*}} Hom(\mathcal{A},\mathcal{A}) \xrightarrow{i^*} 0.$$

To prove that $\mathcal{A}$ is Gorenstein we have to check that

$$Ext^1(\mathbb{C},\mathcal{A}) = \frac{Ker\, M^*}{Im\, \nabla^*} = 0$$

$$Ext^2(\mathbb{C},\mathcal{A}) = \frac{Ker\, \nabla^{t*}}{Im\, M^*} = 0$$

$$Ext^3(\mathbb{C},\mathcal{A}) = \frac{Ker\, i^*}{Im\, \nabla^{t*}} = \mathbb{C}.$$

Let us start by the first of these equalities. Since the sequence

$$\mathcal{A}^{s+1} \xrightarrow{M} \mathcal{A}^{s+1} \xrightarrow{\nabla} Im\,\nabla \to 0$$

where $Im\,\nabla = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$ is exact and $Hom(.,\mathcal{A})$ is a left contravariant functor, we know that

$$0 \to Hom(Im\,\nabla,\mathcal{A}) \xrightarrow{\nabla^*} Hom(\mathcal{A}^{s+1},\mathcal{A}) \xrightarrow{M^*} Hom(\mathcal{A}^{s+1},\mathcal{A})$$

is exact as well. Hence

$$Im\,\nabla^*_B = Ker\, M^*, \tag{8}$$

where $B = Hom(Im\,\nabla,\mathcal{A})$. If $\tilde{f} \in B$, $\tilde{f}$ can be extended to $\mathcal{A}$ by setting $f(a_0) = 0$ if $a_0 \in \mathbb{C}$. Taking into account our definition of $\mathbb{C}$ as $\mathcal{A}$-module one easily checks that f is $\mathcal{A}$-linear. Hence B can be viewed as a subset of $Hom(\mathcal{A},\mathcal{A})$ so that

$$Im\,\nabla^*_B \subset Im\,\nabla^*. \tag{9}$$

On the other hand $\nabla M = 0$ implies $M^*\nabla^* = 0$ and

$$Im\,\nabla^* \subset Ker\, M^*. \tag{10}$$

(8), (9) and (10) imply that $Im\,\nabla^* = Ker\,M^*$ and that $Ext^1(\mathbb{C},\mathcal{A}) = 0$.

Since M is symmetric, we deduce that $Ker\,\nabla^{t*} = Im\,M^*$ and that $Ext^2(\mathbb{C},\mathcal{A}) = 0$. To compute $Ext^3(\mathbb{C},\mathcal{A})$, first note that $0 \equiv f \circ i = i^*(f)$ so that $Ker\,i^* = Hom(\mathcal{A},\mathcal{A})$. If $g \in Hom(\mathcal{A},\mathcal{A})$, g can be written as

$$g = \lambda\epsilon + g_1 + g_2 + \cdots, \tag{11}$$

where $\lambda = g(1) \in \mathbb{C}$ and the g_i are homogeneous of degree i. If $g \in Im\,\nabla^{t*}$, then $g = \nabla^{t*}(h)$, $h \in Hom(\mathcal{A}^{s+1},\mathcal{A})$, and $g(a) = h(a\nabla_0,\ldots,a\nabla_s)$, $\forall a \in \mathcal{A}$. Since $h \in Hom(\mathcal{A}^{s+1},\mathcal{A})$ and $deg\,a\nabla_i \geq 1$, we have that $g(a)$ has no term in degree 0. Then λ in (11) is 0. All this allows us to conclude that

$$Ext^3(\mathbb{C},\mathcal{A}) = \mathbb{C}.$$

and that $\mathcal{A}$ is Gorenstein. □

By Proposition 2.10 one has the following corollary.

Corollary 3.3. *$\mathcal{A}$ has Hochschild dimension 3.*

4 Appendix

As a conclusion of this lecture we will compute the Poincaré series for the Yang-Mills algebra. Since for a N-homogeneous algebra $\mathcal{A} = \oplus_n \mathcal{A}_n$ and the $\mathcal{A}_n$ are all finite dimensional, one can define its Poincaré series as

$$\mathcal{P}_{\mathcal{A}}(t) = \sum_n dim(\mathcal{A}_n)t^n.$$

On the other hand we have constructed the N-complex $K(\mathcal{A}) = \bigoplus_n K_n(\mathcal{A})$, where $K_n(\mathcal{A}) = \mathcal{A} \otimes (\mathcal{A}^!_n)^*$ and $d : K_{n+1}(\mathcal{A}) \to K_n(\mathcal{A})$. Actually one has $d(\mathcal{A}_r \otimes (\mathcal{A}^!_{s+1})^*) \subset d(\mathcal{A}_{r+1} \otimes (\mathcal{A}^!_s)^*)$, and this allows us to split $K(\mathcal{A})$ into sub N-complexes

$$K(\mathcal{A})^n = \oplus_m(\mathcal{A}_{n-m} \otimes (\mathcal{A}^!_m)^*).$$

All this induces a splitting for the Koszul complex $C_{N-1,0} = \mathcal{K}(\mathcal{A},\mathbb{K})$ of $\mathcal{A}$.

$$\mathcal{K}(\mathcal{A},\mathbb{K}) = \oplus_n \mathcal{K}^{(n)},$$

where $\mathcal{K}^{(n)}$ is given by

$$\cdots \xrightarrow{d} \mathcal{A}_{n-N} \otimes (\mathcal{A}^!_N)^* \xrightarrow{d^{N-1}} \mathcal{A}_{n-1} \otimes (\mathcal{A}^!_1)^* \xrightarrow{d} \mathcal{A}_n \to 0.$$

So that the Euler characteristic χ^n of $\mathcal{K}^{(n)}$ can be computed in terms of the dimensions of the $\mathcal{A}_r$ and $\mathcal{A}^!_s$. One has

$$\chi^{(n)} = \sum_k (dim(\mathcal{A}_{n-kN})dim(\mathcal{A}^!_{kN}) - dim(\mathcal{A}_{n-kN-1})dim(\mathcal{A}^!_{kN+1})).$$

Setting

$$\chi_{\mathcal{A}}(t) = \sum_n \chi^{(n)}t^n$$

and

$$Q_{\mathcal{A}}(t) = \sum_n dim(\mathcal{A}^!_{nN})t^{nN} - dim(\mathcal{A}^!_{nN+1})t^{nN+1}$$

one obtains the following formula [DU-PO]

$$\mathcal{P}_{\mathcal{A}}(t)Q_{\mathcal{A}}(t) = \chi_{\mathcal{A}}(t).$$

If $\mathcal{A}$ is Koszul then the complexes $\mathcal{K}^{(n)}$ are acyclic for $n > 0$ and $\chi_{\mathcal{A}} = 1$. In other words, if $\mathcal{A}$ is Koszul

$$\mathcal{P}_{\mathcal{A}}(t) Q_{\mathcal{A}}(t) = 1.$$

In the case of our Yang-Mills algebra

$$Q_{\mathcal{A}}(t) = \sum_n dim(\mathcal{A}^!_{3n}) t^{3n} - dim(\mathcal{A}^!_{3n+1}) t^{3n+1},$$

then

$$Q_{\mathcal{A}}(t) = 1 - (s+1)t + (s+1)t^3 + t^4 = (1-t^2)(1-(s+1)t+t^2)$$

and the corresponding Poincaré series is

$$\mathcal{P}_{\mathcal{A}}(t) = \frac{1}{(1-t^2)(1-(s+1)t+t^2)}.$$

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Higher-Spin Gauge Fields and Duality

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Abstract. We review the construction of free gauge theories for gauge fields in arbitrary representations of the Lorentz group in D dimensions. We describe the multi-form calculus which gives the natural geometric framework for these theories. We also discuss duality transformations that give different field theory representations of the same physical degrees of freedom, and discuss the example of gravity in D dimensions and its dual realisations in detail.

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1 Introduction

Tensor fields in exotic higher-spin representations of the Lorentz group arise as massive modes in string theory, and limits in which such fields might become massless are of particular interest. In such cases, these would have to become higher-spin gauge fields with appropriate gauge invariance. Such exotic gauge fields can also arise as dual representations of more familiar gauge theories [1], [2]. The purpose here is to review the formulation of such exotic gauge theories that was developed in collaboration with Paul de Medeiros in [3], [4].

Free massless particles in D-dimensional Minkowski space are classified by representations of the little group $SO(D-2)$. A bosonic particle is associated with a tensor field $A_{ij\ldots k}$ in some irreducible tensor representation of $SO(D-2)$ and in physical gauge (i.e. in light-cone gauge) the particle is described by a field $A_{ij\ldots k}$ depending on all D coordinates of Minkowski space and satisfying a free wave equation

$$\Box A = 0 \ . \tag{1}$$

For $D=4$, the bosonic representations of the little group $SO(2)$ are classified by an integer, the spin s, while for $D>4$ the representation theory is more involved, although it is common to still refer to generic tensors as being of 'higher spin'.

The main topic to be considered here is the construction of the Lorentz-covariant gauge theory corresponding to these free physical-gauge theories. The first step is finding the appropriate covariant tensor gauge field. For example, an n'th rank antisymmetric tensor physical-gauge field $A_{i_1\ldots i_n} = A_{[i_1\ldots i_n]}$ (where $i,j=1,\ldots,D-2$) arises from a covariant n'th rank antisymmetric tensor gauge field $A_{\mu_1\ldots\mu_n} = A_{[\mu_1\ldots\mu_n]}$ (where $\mu,\nu = 0,1,\ldots,D-1$) with gauge symmetry $\delta A = d\lambda$, while a graviton represented by a traceless symmetric tensor $h_{ij}=h_{ji}$ with $h_i{}^i = 0$ arises from a covariant tensor gauge field $h_{\mu\nu}$ which is symmetric but not traceless, with the usual gauge transformations corresponding to linearised diffeomorphisms. The general rule is to replace an irreducible tensor representation of $SO(D-2)$, given by some tensor field $A_{ij\ldots k}$ with suitable trace-free constraints, by the corresponding tensor field $A_{\mu\nu\ldots\rho}$ with the same symmetry properties as $A_{ij\ldots k}$, but with no constraints on the traces, so that it can be viewed as a tensor representation of $GL(D,\mathbb{R})$. There are some subtleties in this step which we shall return to shortly. The covariant gauge field must transform under gauge symmetries that are sufficient to remove all negative-norm states and to allow the recovery of the physical-gauge theory on gauge fixing.

The next step is the construction of a gauge-invariant field equation and action. For antisymmetric tensors or gravitons, this is straightforward, but for generic higher spin representations the situation is more complicated. One of the simplest cases is that of totally symmetric tensor gauge fields $A_{\mu_1\ldots\mu_n} = A_{(\mu_1\ldots\mu_n)}$. For these, covariant field equations were found by Fronsdal in [5] and reformulated in a geometric language by de Wit and Freedman in [6], but these suffered from the drawback that the gauge fields were constrained, corresponding to a partial fixing of the gauge invariance. This was generalised to arbitrary representations by Siegel and Zwiebach [7], and the duality properties analysed. Covariant field equations and actions have very recently been constructed for totally symmetric tensor gauge fields by Francia and Sagnotti [8], [9] (for a review see the contribution to these proceedings [10]). These have an elegant geometrical structure, being constructed in terms of covariant field strengths, but have the surprising feature of being non-local in general. Nonetheless, on partially fixing the gauge invariance the non-locality is eliminated and the field equations of [5], [6] are recovered. It appears that this non-locality is inescapable in the covariant formulation of higher-spin gauge theories, and it would be interesting to understand whether this has any physical consequences.

Recently, this has been generalised to general higher spin gauge fields in any tensor representation [3], [4], [11], [12]. The formulation of [3], [4] uses an elegant mathematical structure, the multiform calculus, developed in [3], [4] and in [13], [14], [15]. It is the approach of [3], [4] which will be reviewed here. The theory is formulated in terms of covariant field strengths or curvatures, and is non-local but reduces to a local theory on gauge-fixing.

In general, it turns out that a given particle theory corresponding to a particular irreducible tensor representation of $SO(D-2)$ can arise from a number of different covariant field theories, and these covariant field theories are said to give dual realisations of the same theory [1], [2]. For example,

consider an n-form representation of $SO(D-2)$ with field $A_{i_1 \dots i_n}$. This is equivalent to the $\tilde{n}$-form representation, where $\tilde{n} = D - 2 - n$ and so the theory could instead be represented in terms of an $\tilde{n}$-form field $\tilde{A}_{i_1 \dots i_{\tilde{n}}} \equiv \frac{1}{n!}\epsilon_{i_1 \dots i_{\tilde{n}} j_1 \dots j_n} A^{j_1 \dots j_n}$. One can then construct a covariant gauge theory based on an n-form gauge field $A_{\mu_1 \dots \mu_n}$ or a $\tilde{n}$-form gauge field $\tilde{A}_{\mu_1 \dots \mu_{\tilde{n}}}$. These are physically equivalent classically, as they both give equivalent theories in physical gauge. The key feature here is that n-form and $\tilde{n}$-form representations are equivalent for $SO(D-2)$ but distinct for $GL(D, \mathbb{R})$. For the general case, there are a number of distinct representations of $GL(D, \mathbb{R})$ that give rise to equivalent representations of $SO(D-2)$ and so lead to dual formulations of the same physical degrees of freedom. Such dualities [1], [2] can be considered in multi-form gauge theories and in general interchange field equations and Bianchi identities and will also be briefly reviewed here.

2 Young Tableaux

Representations of $GL(D, \mathbb{R})$ can be represented by Young tableaux, with each index μ of a tensor $T_{\mu\nu \dots \rho}$ corresponding to a box in the diagram; see [16] for a full discussion. Symmetrized indices are represented by boxes arranged in a row, so that e.g. a 2nd rank symmetric tensor $h_{\mu\nu}$ is represented by $\ydiagram{2}$, while anti-symmetrized indices are represented by boxes arranged in a column, so that e.g. a 2nd rank anti-symmetric tensor $B_{\mu\nu}$ is represented by $\ydiagram{1,1}$. A general 3rd rank tensor $E_{\mu\nu\rho}$ can be decomposed into a totally symmetric piece $E_{(\mu\nu\rho)}$ represented by the tableau $\ydiagram{3}$, a totally anti-symmetric piece $E_{[\mu\nu\rho]}$ represented by the tableau $\ydiagram{1,1,1}$, and the remaining piece $D_{\mu\nu\rho} \equiv E_{\mu\nu\rho} - E_{(\mu\nu\rho)} - E_{[\mu\nu\rho]}$, which is said to be of mixed symmetry, is represented by the "hook" tableau: $\ydiagram{2,1}$. This satisfies $D_{[\mu\nu\rho]} = 0$ and $D_{(\mu\nu\rho)} = 0$ and is an irreducible representation of $GL(D, \mathbb{R})$. As another example, a fourth-rank tensor $R_{\mu\nu\rho\sigma}$ with the symmetries of the Riemann tensor corresponds to the diagram $\ydiagram{2,2}$.

The same diagrams can be used also to classify representations of $SO(D)$, but with the difference that now all traces must be removed to obtain an irreducible representation. For example, the diagram $\ydiagram{2}$ now regarded as a tableau for $SO(D)$ corresponds to 2nd rank symmetric tensor $h_{\mu\nu}$ that is traceless, $\delta^{\mu\nu} h_{\mu\nu} = 0$. The hook tableau $\ydiagram{2,1}$ now corresponds to a tensor $D_{\mu\nu\rho}$ that is traceless, $\delta^{\mu\rho} D_{\mu\nu\rho} = 0$. Similarly, the diagram $\ydiagram{2,2}$ now corresponds to a tensor with the algebraic properties of the Weyl tensor.

Then given a field in physical gauge in a representation of $SO(D-2)$ corresponding to some Young tableau, the corresponding covariant field in the construction outlined above is in the representation of $GL(D, \mathbb{R})$ corresponding to the *same* Young tableau, now regarded as a tableau for $GL(D, \mathbb{R})$. For example, a graviton is represented in physical gauge by a transverse traceless tensor h_{ij} (with $\delta^{ij} h_{ij} = 0$) of $SO(D-2)$ corresponding to the Young tableau $\ydiagram{2}$, so the covariant formulation is the $GL(D, \mathbb{R})$ representation with tableau $\ydiagram{2}$, which is a symmetric tensor $h_{\mu\nu}$ with no constraints on its trace.

It will be convenient to label tableaux by the lengths of their columns, so that a tableau with columns of length $n_1, n_2, \dots, n_p$ will be said to be of type $[n_1, n_2, \dots, n_p]$. It is conventional to arrange these in decreasing order, $n_1 \geq n_2 \geq \dots \geq n_p$.

3 Duality

Free gauge theories typically have a number of dual formulations. For example, electromagnetism in flat D dimensional space is formulated in terms of a 2-form field strength $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge\, dx^\nu$ satisfying $dF = 0$ and $d * F = 0$, where $*F$ denotes the Hodge dual $D-2$ form with components

$$*F_{\mu_1 \dots \mu_{D-2}} \equiv \frac{1}{2} F^{\nu\rho} \epsilon_{\nu\rho\mu_1 \dots \mu_{D-2}} \; . \tag{2}$$

The equation $dF = 0$ can be interpreted as a Bianchi identity and solved in terms of a 1-form potential A as $F = dA$, with $d * F = 0$ regarded as a field equation for A. Alternatively, one can view $d * F = 0$

as the Bianchi identity $d\tilde{F} = 0$ for $\tilde{F} \equiv *F$, and this implies that $\tilde{F}$ can be written in terms of a $D-3$ form potential $\tilde{A}$ with $\tilde{F} = d\tilde{A}$. Then $dF = 0$ becomes $d * \tilde{F} = 0$ which can be regarded as a field equation for $\tilde{A}$. The theory can be formulated either in terms of the one-form A or in terms of the $D-3$ form potential $\tilde{A}$, giving two dual formulations.

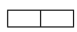
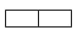
This can be understood from the point of view of the little group $SO(D-2)$. In physical gauge or light-cone gauge, the degrees of freedom are represented by a transverse vector field A_i in the $D-2$ dimensional vector representation of $SO(D-2)$, with $i = 1 \ldots D-2$. This is equivalent to the $(D-3)$-form representation of $SO(D-2)$, so the theory can equivalently be formulated in physical gauge in terms of a $(D-3)$-form

$$\tilde{A}_{j_1 \cdots j_n} = \epsilon_{j_1 \cdots j_n i} A^i \ . \tag{3}$$

where $n = D-3$. These representations of $SO(D-2)$ can be associated with Young tableaux. The vector representation of $SO(D-2)$ is described by a single-box Young tableau,  , while the $(D-3)$-form is associated with a tableau that has one column of $D-3$ boxes. For example in $D=5$, this is a one-column, two-box tableau,  .

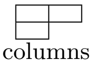
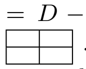
In physical gauge, changing from a 1-form field A_i to a $D-3$ form field $\tilde{A}_{j_1 \cdots j_n}$ is the local field redefinition (3) and so is a trivial rewriting of the theory. However, these lead to two different formulations of the covariant theory: the same physical degrees of freedom can be obtained either from a covariant 1-form gauge field A_μ transforming as a vector under $SO(D-1,1)$, or from a $D-3$ form gauge field $\tilde{A}_{\mu_1 \cdots \mu_n}$. The one-form field has a gauge symmetry $\delta A = d\lambda$ while the $D-3$ form field has a gauge symmetry $\delta \tilde{A} = d\tilde{\lambda}$ and these can be used to eliminate the unphysical degrees of freedom and go to physical gauge. Thus two formulations that are equivalent in physical gauge correspond to two covariant formulations that are distinct covariant realisations of the theory.

This is the key to understanding the generalisations to other gauge fields in other representations of the Lorentz group. A scalar field is a singlet of the little group, and this is equivalent to the $D-2$ form representation of $SO(D-2)$, represented by a tableau with one column consisting of $D-2$ boxes. The scalar field ϕ then has a dual covariant formulation as a $D-2$ form gauge field $\tilde{\phi}_{\mu_1 \cdots \mu_{D-2}}$.

For spin 2, the graviton in D dimensions is a field $h_{\mu\nu}$ which is a symmetric second rank (with trace) represented by the Young tableau  for $GL(D,\mathbb{R})$. The reduction to physical gauge gives a transverse, symmetric, traceless tensor of $SO(D-2)$ h_{ij}, corresponding to the $SO(D-2)$ tableau  . (Recall that for $GL(D,\mathbb{R})$, each box in the tableau represents an index in the D-dimensional representation and has a trace in general, while for $SO(D-2)$ each box in the tableau represents an index in the $(D-2)$-dimensional representation and traces are removed using the $SO(D-2)$ metric, so that the symmetric tensor h_{ij} satisfies $\delta^{ij} h_{ij} = 0$.) The physical gauge graviton h_{ij} can be dualized on one or both of its indices giving respectively

$$D_{i_1 \cdots i_n k} = \epsilon_{i_1 \cdots i_n l} \, h^l{}_k \ , \tag{4}$$

$$C_{i_1 \cdots i_n j_1 \cdots j_n} = \epsilon_{i_1 \cdots i_n l} \, \epsilon_{j_1 \cdots j_n k} \, h^{lk} \ . \tag{5}$$

These give equivalent representations of the little group $SO(D-2)$, with appropriate trace conditions. The tracelessness condition $\delta^{ij} h_{ij} = 0$ implies $D_{[i_1 \cdots i_n k]} = 0$, while the symmetry $h_{[ij]} = 0$ implies the tracelessness $\delta^{i_n k} D_{i_1 \cdots i_n k} = 0$. Then D is represented by the $[n,1]$ hook diagram with one column of length $n = D-3$ and one of length one, so that in dimension $D=5$, D_{ijk} corresponds to the "hook" tableau for $SO(D-2)$:  . The field $C_{i_1 \cdots i_n j_1 \cdots j_n}$ corresponds to the tableau for $GL(D-2,\mathbb{R})$ of type $[n,n]$ with two columns each of $n = D-3$ boxes, so that for $D=5$ C_{ijkl} corresponds to the "window", the two-times-two tableau:  . However, it turns out that $C_{i_1 \cdots i_n j_1 \cdots j_n}$ is not in the $[n,n]$ representation for $SO(D-2)$. In general, the $[m,m]$ representation of $GL(D-2,\mathbb{R})$ would decompose into the representations $[m,m] \oplus [m-1,m-1] \oplus [m-2,m-2] \oplus \ldots.$ of $SO(D-2)$, corresponding to multiple traces. For $m = n = D-3$, it turns out that all the trace-free parts vanish identically, so that only the $[1,1]$ and singlet representations of $SO(D-2)$ survive resulting from $n-1$ and n traces respectively, so that

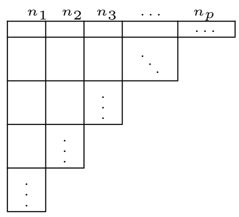
$$C_{i_1 \cdots i_n}{}^{j_1 \cdots j_n} = \delta_{[i_1}{}^{[j_1} \cdots \delta_{i_{n-1}}{}^{j_{n-1}} C_{i_n]}{}^{j_n]} + \delta_{[i_1}{}^{[j_1} \cdots \delta_{i_{n-1}}{}^{j_{n-1}} \delta_{i_n]}{}^{j_n]} C$$

for some C_{ij}, C with traceless C_{ij}. The definition (5) and the tracelessness of h_{ij} then imply that taking n traces of $C_{i_1\cdots i_n j_1\cdots j_n}$ gives zero, so that $C = 0$ and C_{ij} is traceless and in the representation $[1,1]$, and in fact C_{ij} is proportional to h_{ij}.

For arbitrary spin in dimension D the general form for a gauge field in light-cone gauge will be

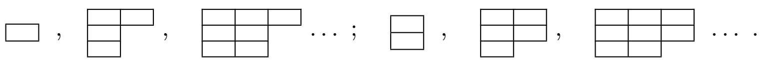
$$D_{[i_1\cdots i_{n_1}][j_1\cdots j_{n_2}]\cdots} \ ,$$

corresponding to an arbitrary representation of the little group $SO(D-2)$, described by a Young tableau with an arbitrary number of columns of lenghts $n_1, n_2\cdots$:



Dual descriptions of such fields can be obtained by dualising any column, i.e. by replacing one of length m with one of length $D-2-m$ (and re-ordering the sequence of columns, if necessary), or by simultaneously dualising a number of columns [2]. Then any of the equivalent physical gauge fields can be covariantised to a gauge field associated with the same tableau, but now viewed as defining a representation of $GL(D,\mathbb{R})$. The set of Young tableaux for these dual representations of the same theory define distinct representations of $GL(D,\mathbb{R})$, but all reduce to equivalent representations of the little group $SO(D-2)$.

In fact, there are yet further dual representations. For $SO(D-2)$, a column of length $D-2$ is a singlet, and given any tableau for $SO(D-2)$, one can obtain yet more dual formulations by adding any number of columns of length $D-2$, then reinterpreting as a tableau for $GL(D,\mathbb{R})$ [7]. Thus for a vector field in $D = 5$, there are dual representations with gauge fields in the representations of $GL(D,\mathbb{R})$ corresponding to the following tableaux:



4 Bi-forms

Before turning to general gauge fields in general representations, we consider the simplest new case, that of gauge fields in representations corresponding to Young tableaux with two columns. It is useful to consider first bi-forms, which are reducible representations in general, arising from the tensor product of two forms, and then at a later stage project onto the irreducible representation corresponding to a Young tableau with two columns. In this section we review the calculus for bi-forms of [3] and generalise to multi-forms and general tableaux in section 7.

A *bi-form* of type (p,q) is an element T of $X^{p,q}$, where $X^{p,q} \equiv \Lambda^p \otimes \Lambda^q$ is the $GL(D,\mathbb{R})$ - reducible tensor product of the space Λ^p of p-forms with the space Λ^q of q-forms on $\mathbb{R}^D$. In components:

$$T = \frac{1}{p!\,q!} T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \otimes dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_q} \ . \tag{6}$$

and is specified by a tensor $T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}$ which is antisymmetric on each of the two sets of p and q indices $T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} = T_{[\mu_1\cdots\mu_p][\nu_1\cdots\nu_q]}$, and no other symmetries are assumed. One can define a number of operations on bi-forms: here we only describe the ones needed for the forthcoming discussion, referring to [3] for a more complete development.

Two exterior derivatives, acting on the two sets of indices, are defined as

$$d : X^{p,q} \to X^{p+1,q} \ , \qquad \textit{left derivative}$$

$$\tilde{d} : X^{p,q} \to X^{p,q+1} \ ,$$ *right derivative* whose action on the elements of $X^{p,q}$ is

$$\begin{aligned} d\,T &= \frac{1}{p!\,q!}\partial_{[\mu}T_{\mu_1\cdots\mu_p]\nu_1\cdots\nu_q}dx^{\mu}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \\ \tilde{d}\,T &= \frac{1}{p!\,q!}\partial_{[\nu}T_{|\mu_1\cdots\mu_p|\nu_1\cdots\nu_q]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \end{aligned} \tag{7}$$

where the two sets of antisymmetric indices are separated by vertical bars. One can verify that

$$d^2 = 0 = \tilde{d}^2, \qquad\qquad d\,\tilde{d} = \tilde{d}\,d \ . \tag{8}$$

With the help of these two exterior derivatives, one can also define the *total derivative*

$$\mathcal{D} \equiv d + \tilde{d} \ , \qquad \text{such that} \qquad \mathcal{D}^3 = 0 \ , \tag{9}$$

where the nilpotency of $\mathcal{D}$ is a straightforward consequence of the nilpotency of d and $\tilde{d}$. Such nilpotent differential operators were considered by [13], [14], [15]. In a similar fashion, restricting to reducible representations of $SO(D-1,1)$, one can introduce two distinct Hodge-duals:

$$\ast : X^{p,q} \to X^{D-p,q} \ , \qquad \textit{left dual}$$
$$\tilde{\ast} : X^{p,q} \to X^{p,D-q} \ , \qquad \textit{right dual}$$

defined as

$$\begin{aligned} \ast\,T &= \frac{1}{p!\,(D-p)!\,q!}T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_D}dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_D}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \\ \tilde{\ast}\,T &= \frac{1}{p!\,q!\,(D-q)!}T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}\epsilon^{\nu_1\cdots\nu_q}{}_{\nu_{q+1}\cdots\nu_D}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_{q+1}}\wedge\cdots\wedge dx^{\nu_D} \ . \end{aligned} \tag{10}$$

These definitions imply that

$$\ast^2 = (-1)^{1+p(D-p)}, \qquad \tilde{\ast}^2 = (-1)^{1+q(D-q)} \ , \qquad \ast\tilde{\ast} = \tilde{\ast}\ast \ , \tag{11}$$

as can be verified recalling the contraction identity for the Ricci-tensor in D dimensions:

$$\epsilon^{\alpha_1\ldots\alpha_k\alpha_{k+1}\ldots\alpha_D}\epsilon_{\alpha_1\ldots\alpha_k\beta_{k+1}\ldots\beta_D} \ = \ -(D-k)!\cdot(\delta^{[\alpha_{k+1}}_{\beta_{k+1}}\ldots\delta^{\alpha_D]}_{\beta_D}) \ , \tag{12}$$

where we are using the "mostly plus" flat background metric.

There are three operations on bi-forms that enter the Bianchi identities and the equations of motion, and into the projections onto irreducible representations: a trace, a dual trace, and a transposition.

A trace operator acts on a pair of indices belonging to different sets, so that

$$\tau : X^{p,q} \to X^{p-1,q-1} \ ,$$

and is defined by

$$\tau\,T \equiv \frac{1}{(p-1)!\,(q-1)!}\eta^{\mu_1\nu_1}T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_2}\wedge\cdots\wedge dx^{\nu_q} \ . \tag{13}$$

Combining the τ operator with the Hodge duals, one can also define two distinct dual traces:

$$\begin{aligned} \sigma &\equiv (-1)^{1+D(p+1)}\ast\tau\ast : X^{p,q} \to X^{p+1,q-1} \ , \\ \tilde{\sigma} &\equiv (-1)^{1+D(q+1)}\tilde{\ast}\tau\tilde{\ast} : X^{p,q} \to X^{p-1,q+1} \ , \end{aligned} \tag{14}$$

that antisymmetrize one index in a set with respect to the whole other set:

$$\begin{aligned} \sigma\,T &= \frac{(-1)^{p+1}}{p!\,(q-1)!}T_{[\mu_1\cdots\mu_p\nu_1]\cdots\nu_q}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ , \\ \tilde{\sigma}\,T &= \frac{(-1)^{q+1}}{(p-1)!\,q!}T_{\mu_1\cdots[\mu_p\nu_1\cdots\nu_q]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q} \ . \end{aligned} \tag{15}$$

Again, the proof of (15) relays on the identity (12). For example, for a $(2,3)$ form in $D=5$ one has (omitting combinatorial factors):

$$
\begin{aligned}
*\,T &\sim T_{p_1p_2,q_1q_2q_3}\epsilon^{p_1p_2}{}_{p_3p_4p_5}\ , \\
\tau*T &\sim T_{p_1p_2,q_1q_2q_3}\epsilon^{p_1p_2q_1}{}_{p_4p_5}\ , \\
*\tau*T &\sim T_{p_1p_2,q_1q_2q_3}\epsilon^{p_1p_2q_1}{}_{p_4p_5}\epsilon^{p_4p_5}{}_{\alpha_1\alpha_1\alpha_3} \sim T_{p_1p_2,q_1q_2q_3}\delta^{[p_1p_2q_1]}_{\alpha_1\alpha_2\alpha_3}\ .
\end{aligned}
\tag{16}
$$

Finally, the transposition operator simply interchanges the two sets of indices:

$$t : X^{p,q} \to X^{q,p}\ ,$$

so that

$$(t\,T)_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p} = T_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q}$$

and

$$t\,T \equiv \frac{1}{p!q!}T_{\nu_1\cdots\nu_q\mu_1\cdots\mu_p}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}\otimes dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\ . \tag{17}$$

The bi-forms are a reducible representation of $GL(D,\mathbb{R})$. It is useful to introduce the Young symmetrizer $\mathcal{Y}_{[p,q]}$ which projects a bi-form T of type (p,q) onto the part $\hat{T}=\mathcal{Y}_{[p,q]}\,T$ lying in the irreducible representation corresponding to a tableau of type $[p,q]$, with two columns of length p and q, respectively (we use round brackets for reducible (p,q) bi-forms and square ones for irreducible representations). The projected part $\hat{T}$ satisfies the additional constraints (for $p\geq q$):

$$
\begin{aligned}
&\sigma\,\hat{T}=0\ , \\
&t\,\hat{T}=\hat{T}, \qquad \text{if} \quad p=q\ .
\end{aligned}
\tag{18}
$$

5 D-Dimensional Linearised Gravity

It is straightforward to formulate gauge field theories of bi-forms; a gauge field A in the space $X^{p,q}$ can be thought of as a linear combination of terms arising from the tensor product of a p-form gauge field and a q-form gauge field. It transforms under the gauge transformation

$$\delta\,A = d\,\alpha^{\,p-1,q}+\tilde{d}\,\alpha^{\,p,q-1}, \tag{19}$$

with gauge parameters that are themselves bi-forms in $X^{\,p-1,q}$, $X^{\,p,q-1}$. Clearly,

$$F = d\,\tilde{d}\,A \tag{20}$$

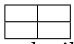
is a gauge invariant field-strength for A. This is a convenient starting point for describing gauge fields in irreducible representations. We now show how to project the bi-form gauge theory using Young projections to obtain irreducible gauge theories, starting with one of the simplest examples, that of linearised gravity in D dimensions.

The graviton field is a rank-two tensor in an irreducible representation of $GL(D,\mathbb{R})$ described by a Young tableau of type $[1,1]$, i.e. a two-column, one-row Young tableau, $\square\!\square$. The starting point in our case is thus a bi-form $h\in X^{1,1}$, corresponding to a 2nd rank tensor $h_{\mu\nu}$, and we would like to project on the $GL(D,\mathbb{R})-$irreducible tensor of type $[1,1]$: $\widehat{h}=\mathcal{Y}_{[1,1]}\,h$ using the Young projector $\mathcal{Y}_{[1,1]}$. Then the constraints (18) become

$$
\begin{aligned}
&\sigma\,\widehat{h}=0\ , \\
&t\,\widehat{h}=\widehat{h}\ .
\end{aligned}
\tag{21}
$$

In this case, the two conditions are equivalent and simply imply that $\widehat{h}$ is symmetric, $\widehat{h}_{\mu\nu} = h_{(\mu\nu)}$. The gauge transformation for the graviton is the Young-projection of (19)[1], which gives $\delta\,\widehat{h}_{\mu\nu} = \partial_{(\mu}\lambda_{\nu)}$ where $\lambda_\mu = \alpha_\mu^{1,0} + \alpha_\mu^{0,1}$. The invariant field strength is given by [2]

$$R = \tilde{d}\,d\,\widehat{h}\ . \tag{22}$$

This is the $[2,2]$ Young tableau  describing the linearized Riemann tensor. The nilpotency of the exterior derivatives and the irreducibility imply that the Bianchi identities

$$d\,R = 0\ , \qquad\qquad \tilde{d}\,R = 0\ , \tag{23}$$

$$\sigma\,R = 0\ , \tag{24}$$

are satisfied, while acting with the τ operator gives the Einstein equation in $D \geq 4$ [3] :

$$\tau\,R = 0\ . \tag{25}$$

or in components, $R_{\mu\nu} = 0$.

We now return to the issue of duality. In Section 3 we described the triality of linearised gravity in D dimensions, for which there are three different fields that can be used for describing the degrees of freedom of the graviton. The discussion can be expressed succinctly in terms of bi-forms. In light-cone gauge the fields are tensors in irreducible representations of $SO(D-2)$, and so are trace-less. The graviton arises from projecting a $[1,1]$ form h onto a symmetric tensor $\widehat{h}_{ij}$ that is traceless $\widehat{h}^i{}_i = 0$. Now one can easily dualise the field $\widehat{h}$ in one or both indices, by applying (10), where the $*$-operator is now the $SO(D-2)$-covariant dual. The dual light-cone fields are

$$D \quad = *\,\widehat{h}\ , \tag{26}$$

$$C \quad = *\,\tilde{*}\,\widehat{h}\ , \tag{27}$$

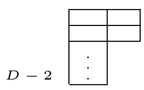
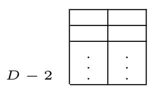
and all have the same number of independent components.

In the covariant theory one dualises the field strengths rather than the gauge fields and this is easily analysed using the bi-form formalism developed. Indeed, starting from the $[2,2]$ field strength R one can define the Hodge duals

$$S \quad \equiv *\,R, \tag{28}$$

$$G \quad \equiv *\,\tilde{*}\,R, \tag{29}$$

which are respectively of type $[D-2,2]$ and $[D-2,D-2]$, associated with the tableaux:

 and  .

The other possible dual, $\tilde{S} \equiv \tilde{*}\,R$ is not independent, since $\tilde{S} = t\,S$ (this would not be the case for the generalisation to a general (p,q)-form with $p \neq q$). In components:

$$S_{\mu_1\cdots\mu_{D-2}\nu_1\nu_2} \quad = \quad \frac{1}{2} R^{\alpha\beta}{}_{\nu_1\nu_2}\,\epsilon_{\alpha\beta\mu_1\cdots\mu_{D-2}}\ , \tag{30}$$

$$G_{\mu_1\cdots\mu_{D-2}\nu_1\cdots\nu_{D-2}} \quad = \quad \frac{1}{4} R^{\alpha\beta\gamma\delta}\,\epsilon_{\alpha\beta\mu_1\cdots\mu_{D-2}}\,\epsilon_{\gamma\delta\nu_1\cdots\nu_{D-2}}\ . \tag{31}$$

[1] Note that acting on a tensor in an irreducible representation with d or $\tilde{d}$ gives a *reducible* form in general, so that a Young projection is necessary in order to obtain irreducible tensors.

[2] The operator $d\,\tilde{d}$, unlike d and $\tilde{d}$ separately, sends irreps to irreps, so that $\tilde{d}\,d\,\widehat{h} = \mathcal{Y}_{[1,1]}\,\tilde{d}\,d\,\widehat{h}$ and the Young projection is automatically implemented.

[3] In $D = 3$ the field equation $R_{\mu\nu} = 0$ implies $R_{\mu\nu\rho\sigma} = 0$ which only has trivial solutions; a non trivial equation is instead $\tau^2\,R = 0$, with $\tau^2\,R$ the Ricci-scalar [2], [1]. This can be generalized to (p,q)-forms, as we shall see later.

From these definitions, and the algebraic and dynamical constraints satisfied by the linearised Riemann tensor R, one can deduce a set of relations that must be obeyed by the bi-forms S and G. We will give examples of certain relations between Bianchi identities and equations of motion, referring to [3], [2], [1] for a more complete discussion. The definitions $S \equiv *R$ and $G \equiv *\tilde{*}R$, and the relations (11) imply that $*S = (-1)^D R$, and $\tilde{*} * G = R$. Then, using the definitions given in Section 4, it follows that

$$\sigma R = 0 \quad \Rightarrow \quad \sigma * S = 0 \quad \Rightarrow \quad *\sigma * S = 0 \quad \Rightarrow \quad \tau S = 0 \; ; \tag{32}$$

$$\sigma R = 0 \;\; \Rightarrow \;\; \sigma \tilde{*} * G = 0 \;\; \Rightarrow \;\; *\sigma \tilde{*} * G = 0 \;\; \Rightarrow \;\; \tau \tilde{*} G = 0 \;\; \Rightarrow \;\; \tilde{*}\tau\tilde{*} G = 0 \;\; \Rightarrow \;\; \tilde{\sigma} G = 0 \; . \tag{33}$$

That is to say, the Bianchi identity $\sigma R = 0$ for R implies the equation of motion $\tau S = 0$ for S and the Bianchi identity $\tilde{\sigma} G = 0$ for G. The equation of motion $\tau R = 0$ for R in $D > 3$ implies that

$$\tau R = 0 \quad \Rightarrow \quad *\tau * S = 0 \quad \Rightarrow \quad \sigma S = 0 \; , \tag{34}$$

and

$$\tau R = 0 \quad \Rightarrow \quad \tau \tilde{*} * G = 0 \quad \Rightarrow \quad \tau^{D-3} G = 0 \; . \tag{35}$$

giving the Bianchi identity $\sigma S = 0$ for S and the field equation $\tau^{D-3} G = 0$ for G [4].

Other consequences for S and G can be deduced starting from properties of R and making use of identities involving the various bi-form operators (see [3]). In particular the Bianchi identities $dS = \tilde{d}S = 0$ and $dG = \tilde{d}G = 0$ imply that S and G can be expressed as field-strengths of gauge potentials $\widehat{D}$ and $\widehat{C}$ respectively, which are in irreducible representations of type $[D-3, 1]$ and $[D-3, D-3]$

$$S = d\,\tilde{d}\,\widehat{D}, \qquad G = d\,\tilde{d}\,\widehat{C} \; , \tag{36}$$

whose linearized equations of motion are $\tau S = 0$ and $\tau^{D-3} G = 0$. Although these relations can be derived for gravity straightforwardly, as in [2], the bi-form formalism simplifies the discussion and generalises to general multi-form representations in a way that elucidates the geometric structure and allows simple derivations and calculations.

6 General Bi-Form Gauge Theories

The discussion of gravity extends straightforwardly to arbitrary (p, q)-forms, where without loss of generality we assume $p \geq q$. First, one can restrict from a (p, q)-form T to $\widehat{T} = \mathcal{Y}_{[p,q]}\, T$ which is in $[p, q]$ irrep of $GL(D, \mathbb{R})$ satisfying the constraints (18). Then one can define a field strength $F \equiv \tilde{d}\, d\, \widehat{T}$ of type $[p+1, q+1]$ that is invariant under the gauge transformations given by the projection of (19):

$$\delta \widehat{T} = \mathcal{Y}_{[p,q]}\, (d\, \alpha^{\, p-1, q} + \tilde{d}\, \alpha^{\, p, q-1}), \tag{37}$$

and satisfies the Bianchi identities

$$dF = \tilde{d}F = 0 \; , \qquad \sigma F = 0 \; , \tag{38}$$

together with $tF = F$ if $p = q$. We now turn to the generalisation of the "Einstein equation" $\tau R = 0$. The natural guess is

$$\tau F = 0 \; , \tag{39}$$

However, we have seen that for gravity in $D = 3$, the Einstein equation $\tau R = 0$ is too strong and only has trivial solutions, but that the weaker condition $\tau^2 R = 0$ (requiring that the Ricci scalar is zero) gives a non-trivial theory. For the dual field strength G, the field equation was $\tau^{D-3} G = 0$ in D dimensions. Then it is to be expected that the "Einstein equation" $\tau R = 0$ will be generalised to $[p, q]$ forms by taking $\tau F = 0$ for large enough space-time dimension D, but for low D a number of traces

[4] Note that the equation $\tau^n G = 0$ only has trivial solutions for $n < D-3$, so that this is the simplest non-trivial field equation [2].

of the field strength may be needed to give an equation of motion $\tau^n\, F = 0$ for some n. It was shown in [3] that the natural equation of motion

$$\tau\, F = 0 \qquad \text{for} \qquad D \geq p + q + 2 \ , \tag{40}$$

is non trivial for $D \geq p + q + 2$, but for $D < p + q + 2$ that $\tau^n\, F = 0$ is a non-trivial field equation for $n = p + q + 3 - D$, and so we will take

$$\tau^{p+q+3-D}\, F = 0 \qquad \text{for} \qquad D < p + q + 2 \ . \tag{41}$$

A $[p, q]$-Young tableau can be dualized on one of the two columns, or on both, so that three duals of the field strength can be defined:

$$S \equiv *\, F \in X^{D-p-1,q+1} \ , \qquad \tilde{S} \equiv \tilde{*}\, F \in X^{p+1,D-q-1} \ , \qquad G \equiv \tilde{*}\, *\, F \in X^{D-p-1,D-q-1} \ , \tag{42}$$

and the algebraic and differential identities and equations of motion for F give analogous properties for S, $\tilde{S}$ and G. In particular, the equations of motion are [5]:

$$\tau\, S = 0 \ , \qquad \tau^{1+p-q}\, \tilde{S} = 0 \ , \qquad \tau^{D-p-q-2+n}\, G = 0 \ . \tag{43}$$

For example, gravity in $D = 3$ with $p = q = 1$ has dual formulations in terms of a $[1, 1]$ field strength G for a $[0, 0]$ form or scalar field C with field equation $\tau\, G = 0$ giving the usual scalar field equation $\partial_\mu\, \partial^\mu\, C = 0$, or to a $[2, 1]$ field strength S for a $[1, 0]$ or vector gauge field D_μ, with the usual Maxwell equation $\tau\, S = 0$. Then this $D = 3$ gravity theory is dual to a scalar field and to a vector field, and all describe one physical degree of freedom.

7 Multi-Forms

The previous discussion generalizes to the case of fields in arbitrary massless representations of $SO(D-1, 1)$, including higher-spin gauge fields described by mixed-symmetry Young tableaux. As for the case of forms and bi-forms, the starting point is the definition of a larger environment, the space of *multi-forms*, in which a series of useful operations are easily defined. Then, by suitable Young projections, one can discuss the cases of irreducible gauge fields and their duality properties. In the following we shall confine ourselves to describing the main steps of the construction; further details are given in [3], [4].

A multi-form of order N is characterised by a set of N integers $(p_1, p_2, ..., p_N)$ and is a tensor of rank $\sum p_i$ whose components

$$T_{\mu_1^1 \cdots \mu_{p_1}^1 \cdots \mu_1^N \cdots \mu_{p_N}^N} = T_{[\mu_1^1 \cdots \mu_{p_1}^1] \cdots [\mu_1^N \cdots \mu_{p_N}^N]} \ , \tag{44}$$

are totally antisymmetrized within each of N groups of p_i indices, with no other symmetry *a priori* between indices belonging to different sets. It is an element of $X^{p_1 \cdots p_N} \equiv \Lambda^{p_1} \otimes \cdots \otimes \ \Lambda^{p_N}$, the $GL(D, \mathbb{R})$-reducible N-fold tensor product space of p_i-forms on $\mathbb{R}^D$. The operations and the properties introduced in Section 4 generalize easily to multi-forms. For an extensive treatment see again [3]; here we restrict our attention to the operations previously discussed.

One can define an exterior derivative acting on the $i - th$ set of indices,

$$d^{(i)} : X^{p_1 \cdots p_i \cdots p_N} \rightarrow X^{p_1 \cdots p_i + 1 \cdots p_N} \ , \tag{45}$$

generalizing the properties of d and $\tilde{d}$; summing over the $d^{(i)}$'s one can then define the *total derivative*

$$\mathcal{D} \equiv \sum_{i=1}^{N} d^{(i)} \ , \tag{46}$$

such that

$$\mathcal{D}^{N+1} = 0 \ . \tag{47}$$

[5]The last equation follows from the result $\tau^n\, T = 0 \Rightarrow (\tau^{D-p-q+n}\, *\, \tilde{*})\, T = 0$, valid for a general [p,q]-form T, applied to the case of the [p+1,q+1]-form F. Here n is the exponent such that $\tau^n\, F = 0$ is non trivial. So, if $D \geq p + q + 2$ then $n = 1$, and the e.o.m. for the dual tensor G reduces to $\tau^{D-p-q-1}\, G = 0$ [3] .

Similarly, for representations of $SO(D-1,1)$ or $SO(D)$ one can define N Hodge-duals:

$$*^{(i)} : X^{p_1,\ldots,p_i,\ldots,p_N} \to X^{p_1,\ldots,D-p_i,\ldots,p_N} \ , \tag{48}$$

each acting in the usual fashion on the i-th form, and so commuting with any $*^{(j\neq i)}$.

The operators τ, σ, $\tilde{\sigma}$ and t generalize to a set of operators, each acting on a specific pair of indices; the *trace* operators

$$\tau^{(ij)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_{i-1},\ldots,p_{j-1},\ldots,p_N} \ , \tag{49}$$

are defined as traces over the i-th and the j-th set; the *dual-traces* are

$$\begin{aligned}\sigma^{(ij)} &\equiv (-1)^{1+D(p_i+1)} *^{(i)} \tau^{(ij)} *^{(i)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_{i+1},\ldots,p_{j-1},\cdots p_N} \ ,\\ \tilde{\sigma}^{(ij)} &\equiv (-1)^{1+D(p_j+1)} *^{(j)} \tau^{(ij)} *^{(j)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_{i-1},\ldots,p_{j+1},\cdots p_N} \ ,\end{aligned} \tag{50}$$

while the *transpositions* $t^{(ij)}$ generalize the action of the t operator to exchanges between the subspaces Λ^{p_i} and Λ^{p_j} in $X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N}$:

$$t^{(ij)} : X^{p_1,\ldots,p_i,\ldots,p_j,\ldots,p_N} \to X^{p_1,\ldots,p_j,\ldots,p_i,\ldots,p_N} . \tag{51}$$

The Young symmetrizer $\mathcal{Y}_{[p_1,\ldots,p_N]}$ projects a multi-form of type $(p_1,\ldots,p_N)$ onto the irreducible representation associated with a Young tableau of type $[p_1,\ldots,p_N]$.

8 Multi-Form Gauge Theories

With the machinery of the last section, one can naturally extend the construction of gauge theories for general tensor gauge fields. The starting point is a multi-form gauge field of type $(p_1,\ldots,p_N)$ with gauge transformation

$$\delta T = \sum_{i=1}^{N} d^{(i)} \, \alpha_{(i)}^{p_1,\ldots,p_{i-1},\ldots,p_N} \ . \tag{52}$$

The restriction to irreducible representations of $GL(D,\mathbb{R})$ can be implemented using the Young symmetrizer $\mathcal{Y}_{[p_1,\ldots,p_N]}$ projecting onto the representation characterised by a Young tableau with N columns of length $p_1, p_2, \ldots, p_N$ (these are conventionally arranged in order of decreasing length, but this is not essential here). Then this projects a multi-form T onto

$$\widehat{T} = \mathcal{Y}_{[p_1,\ldots,p_N]} \, T \tag{53}$$

which satisfies the constraints

$$\begin{aligned}\sigma^{ij}\, \widehat{T} &= 0 \qquad \text{if} \quad p_i \geq p_j\\ t^{ij}\, \widehat{T} &= \widehat{T} \qquad \text{if} \quad p_i = p_j\end{aligned} \tag{54}$$

The field strength is a multi-form in the irreducible representation of type $[p_1+1,\ldots,p_N+1]$ defined as[6]

$$F \equiv \prod_{i=1}^{N} d^{(i)}\, \widehat{T} = \frac{1}{N} \mathcal{D}^N\, \widehat{T} \ , \tag{55}$$

and is invariant under the Young projection of the gauge transformation for T

$$\delta\, \widehat{T} = \mathcal{Y}_{[p_1,\ldots,p_N]} \sum_{i=1}^{N} d^{(i)} \, \alpha_{(i)}^{p_1,\ldots,p_{i-1},\ldots,p_N} \ . \tag{56}$$

[6]More generally, one can define a set of connections $\Gamma_{S_k} \equiv (\prod_{i\notin S_k} d^{(i)})\, \widehat{T}$, corresponding to each subset $S_k = \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,N\}$. These are gauge-dependent w.r.t. transformations involving parameters α^j , $j \in S_k$, while are invariant under transformations with parameters α^i , $i \notin S_k$. For a given k there are in general $\frac{N!}{(N-k)!k!}$ inequivalent possible Γ_{S_k}; in particular, the totally gauge-invariant field strenght F can be regarded both as the top of this hierarchy of connections (the one with $k=0$), or as a direct function of the connection Γ_{S_k}, being $F = (\prod_{i\in S_k} d^{(i)})\, \Gamma_{S_k}$ [4] .

By construction, the field strength satisfies the generalized Bianchi identities

$$d^{(i)}\,F = 0\ , \tag{57}$$
$$\sigma^{(ij)}\,F = 0\ . \tag{58}$$

The simplest covariant local field equations are those proposed in [3] and these in general involve more than two derivatives. For N even, a suitable field equation is

$$\sum_{\substack{(i_1,\dots,i_N)\\(permutations)}} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)}\,F = 0\ , \tag{59}$$

where the sum is over all permutations of the elements of the set $\{1,\dots N\}$. For N odd, one first needs to define $\partial\,F$, which is the derivative $\partial_\mu\,F$ of F regarded as a rank $N+1$ multi-form of type $[p_1,\dots,p_N,1]$. Then the equation of motion is:

$$\sum_{\substack{(i_1,\dots,i_N)\\(permutations)}} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-2} i_{N-1})}\tau^{(i_N N+1)}\,\partial\,F = 0\ . \tag{60}$$

Here the sum is over *the same* set of permutations of the elements of the set $\{1,\dots N\}$ as in the even case, so that the extra index is left out. These are the field equations for large enough space-time dimension D; as for the case of bi-forms, for low dimensions one needs to act with further traces.

These field equations involving multiple traces of a higher-derivative tensor are necessarily of higher order in derivatives if $N>2$. This is unavoidable if the field equation for a higher-spin field is to be written in terms of invariant curvatures. In physical gauge, these field equations become $\Box^a A = 0$ where A is the gauge potential in physical gauge, $\Box$ is the D-dimensional d'Alembertian operator and $a = N/2$ if N is even and $a = (N+1)/2$ if N is odd. The full covariant field equation is of order $2a$ in derivatives. In order to get a second order equation, following [8, 9], one can act on these covariant field equations with $\Box^{1-a}$ to obtain equations that reduce to the second order equation $\Box A = 0$ in physical gauge. In the even case the equation (59) is of order N in derivatives, and so it is possible to write a second-order field equation dividing by $\Box^{\frac{N}{2}-1}$. Similarly, for N odd, it is necessary to divide by $\Box^{\frac{N+1}{2}-1}$. In this way, one can write second-order, non-local field equations [4]:

$$\begin{aligned}
\mathcal{G}_{\text{even}} &\equiv \sum_{\substack{(i_1,\dots,i_N)\\(permutations)}} \tau^{(i_1 i_2)} \dots \tau^{(i_{N-1} i_N)}\,\frac{1}{\Box^{\frac{N}{2}-1}}\,F = 0\ ,\\
\mathcal{G}_{\text{odd}} &\equiv \sum_{\substack{(i_1,\dots,i_N)\\(permutations)}} \tau^{(i_1 i_2)} \dots \tau^{(i_N N+1)}\,\frac{1}{\Box^{\frac{N+1}{2}-1}}\,\partial\,F = 0\ .
\end{aligned} \tag{61}$$

These then are the covariant field equations for general representations for high enough D (for low D, the appropriate field equations require further traces [4], as we saw earlier for the case of bi-forms.) These are non-local, but after fixing a suitable gauge, they become local. On fully fixing the gauge symmetry to go to light-cone gauge, the field equations reduce to the free equation $\Box A = 0$, while partially fixing the gauge gives a Fronsdal-like local covariant field equation with constraints on the traces of the gauge field and parameters of the surviving gauge symmetries. It would be interesting to understand if the non-locality of the full geometric field equation has any physical consequences, or is purely a gauge artifact. As in the Fronsdal case, only physical polarizations are propagating [9, 4, 12].

It is worth noting that these equations are not unique. As was observed in [8], and analysed in detail for the case $s=3$ in the totally symmetric representation, one can write other second-order equations, with higher degree of non locality, by combining the least singular non-local equation with its traces and divergences. The systematics of this phenomenon was described in [4], where it was shown in the general case how to generate other field equations starting from (61). The idea is to define a new tensor $F^{(m)} \equiv \partial^m F$, by taking m partial derivatives of the field strenght F, take a suitable number of traces

of the order $N+m$ tensor $\partial^m F$, and divide for the right power of the D'Alembertian operator. One can then take linear combinations of these equations with the original equations (61).

Given a field strength F of type $[p_1+1, p_2+1, \ldots, p_N+1]$, one can choose any set of columns of the Young tableau and dualise on them to obtain a dual field strength. The field equations and Bianchi identities for F then give the field equations and Bianchi identities for the dual field strength, and the new Bianchi identities imply that the dual field strength can be solved for in terms of a dual potential. There are then many dual descriptions of the same free higher-spin gauge theory.

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Higher spin fields from indefinite Kac–Moody algebras

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Abstract. The emergence of higher spin fields in the Kac–Moody theoretic approach to M-theory is studied. This is based on work done by Schnakenburg, West and the second author. We then study the relation of higher spin fields in this approach to other results in different constructions of higher spin field dynamics. Of particular interest is the construction of space-time in the present set-up and we comment on the various existing proposals.

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1 Introduction and Motivation

M-theory is usually thought to comprise the different string theories and is therefore related to higher spin (HS) field theories in the sense that string theories contain HS. Indeed, the perturbative spectra of the different string theories characteristically contain massive HS fields if the tension of the string is finite and massless HS fields in the limit of vanishing tension. The tensionless limit $\alpha' \to \infty$, in which all perturbative states become massless, is of particular interest because of the restoration of very large symmetries [1]. One possible approach to M-theory is based on such infinite-dimensional symmetries; work along these lines can be found in [2, 3, 4, 5, 6] and references therein. Our focus here will be the Kac–Moody theoretical approach of [3] and [5, 6]. The aim of the present discussion is to show how higher spin fields arise naturally as objects in this algebraic formulation and speculate how a dynamical scheme for these fields might emerge. Before studying the details of the HS fields, let us briefly review the origin of the Kac–Moody algebras we are going to consider.

It is a longstanding conjecture that the reduction of certain (super-)gravity systems to very low space-time dimensions exhibits infinite-dimensional symmetry algebras [7]. As these sectors are normally thought of as low energy limits of M-theory, these conjectures extend to M-theory, albeit with certain modifications [8]. The precise statement for the (super-)gravity systems is that the scalar sector (with dualisation of fields to scalar fields whenever possible) is described by a non-linear sigma model $G/K(G)$ where G is some group and $K(G)$ its maximal compact subgroup [7].

As an example, the chain of these so-called 'hidden symmetries' in the case of $D_{max} = 11$, $N = 1$ supergravity is displayed in the table below. The scalar sector of this theory reduced to D space-time dimensions is described by a scalar coset model $G/K(G)$ (G is in the split form) determined by

D	G	K
9	$SL(2,R) \times SO(1,1)$	$SO(2)$
8	$S(3,R) \times SL(2,R)$	$U(2)$
7	$SL(5,R)$	$USp(4)$
6	$SO(5,5)$	$USp(4) \times USp(4)$
5	$E_{6(6)}$	$USp(8)$
4	$E_{7(7)}$	$SU(8)$
3	$E_{8(8)}$	$Spin(16)/Z_2$
2	$E_{9(9)}$	$K(E_9)$
1	$E_{10(10)}$	$K(E_{10})$
(0	$E_{11(11)}$	$K(E_{11})$)

After the reduction to three dimensions one obtains Kac–Moody theoretic extensions of the exceptional E-series. The two-dimensional symmetry E_9 is an affine symmetry [7, 9], below two dimensions the expected symmetry is the hyperbolic extension E_{10} and in one dimension one formally finds the Lorentzian algebra E_{11}. Three dimensions are special since there all physical (bosonic) degrees of freedom can be converted into scalars. The procedure inverse to dimensional reduction, called oxidation, of a theory has also been studied [10, 11, 12]. There one starts with a scalar coset model G/K in three space-time dimensions for each simple G and asks for a higher-dimensional theory whose reduction yields this hidden symmetry. The answer is known for all semi-simple G and was presented in [11] and will be recovered in table 4.1 below. For conciseness of notation we denote the oxidized theory in the maximal dimension by $\mathcal{O}_G$. In this language, maximal eleven-dimensional supergravity is denoted by $\mathcal{O}_{E_8}$.

A different motivation for studying infinite-dimensional Kac–Moody algebras comes from cosmological billiards. Indeed, it has been shown recently that the dynamics of the gravitational scale factors becomes equivalent, in the vicinity of a spacelike singularity, to that of a relativistic particle moving freely on an hyperbolic billiard and bouncing off its walls [13, 14, 15, 16]. A criterion for the gravitational dynamics to be chaotic is that the billiard has a finite volume [17]. This in turn stems from the remarkable property that the billiard walls can be identified with the walls of the fundamental Weyl

chamber of a hyperbolic Kac–Moody algebra.[1] Building on these observations it has been shown that a null geodesic motion of a relativistic particle in the coset space $E_{10}/K(E_{10})$ can be mapped to the bosonic dynamics of $D = 11$ supergravity reduced to one time dimension [5, 15].

It is at the heart of the proposal of [3] that the hidden symmetries of the reduced theory are already present in the unreduced theory. Furthermore, the symmetry groups actually get extended from the finite-dimensional G to the Lorentzian triple extension G^{+++} (also called very-extension).[2] To every finite-dimensional G we wish to associate a model possessing a non-linearly realized G^{+++} symmetry which might be called the M-theory corresponding to G, and denoted by $\mathcal{V}_G$. To try to construct M-theory in the sense given above, the basic tool is a non-linear sigma model based on Kac–Moody algebras in the maximal number of dimensions (corresponding to the oxidized $\mathcal{O}_G$). Basic features about non-linear sigma models associated with the coset $G/K(G)$ are recalled in appendix A. M-theories constructed in this way possess an infinite-dimensional symmetry structure and contain infinitely many fields. Some of the infinitely many fields may be auxiliary, however.[3] In addition one can hope to relate some of the infinitely many other fields to the perturbative string spectrum. Since the underlying Kac–Moody algebra is not well-understood the analysis is complicated and incomplete, and this identification of states has not been carried out. Below we will present the result for the supergravity fields. In addition, the precise dynamical theory for all fields is mostly elusive to date. There remain a number of important challenges in the full understanding of these M-theories, which will be highlighted at the relevant points.

Since E_{11} will be our guiding example, we briefly review the evidence for the conjecture that M-theory possesses an E_{11} symmetry.

- It has been shown that the bosonic sector of $D = 11$ supergravity can be formulated as the simultaneous non-linear realization of two finite-dimensional Lie algebras [21]. The two corresponding groups, whose closure is taken, are the eleven-dimensional conformal group and a group called G_{11} in [21]. The generators of G_{11} contain the generators P_a and $K^a{}_b$ of the group of affine coordinate transformations $IGL(11)$ in eleven dimensions and the closure with the conformal group will therefore generate infinitesimal general coordinate transformations [22]. The precise structure of the Lie algebra of G_{11} is given by

$$\begin{aligned}
[K^a{}_b, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b \\
[K^a{}_b, P_c] &= -\delta^a_c P_b \\
[K^a{}_b, R^{c_1\cdots c_6}] &= -6\delta_b^{[c_1} R^{c_2\cdots c_6]a}, \\
[K^a{}_b, R^{c_1\cdots c_3}] &= 3\delta_b^{[c_1} R^{c_2 c_3]a}, \\
[R^{c_1\cdots c_3}, R^{c_4\cdots c_6}] &= 2R^{c_1\cdots c_6}.
\end{aligned} \tag{1}$$

The additional totally anti-symmetric generators $R^{c_1\cdots c_3}$ and $R^{c_1\cdots c_6}$ are obviously related to the three-form gauge potential of $N = 1$, $D = 11$ supergravity and its (magnetic) dual. To write the equations of motion of the bosonic sector of $D = 11$ supergravity one should first consider the following coset element of G_{11} over the Lorentz group H_{11},

$$v(x) = e^{x^a P_a} e^{h_a{}^b K^a{}_b} \exp\left(\frac{A_{c_1\dots c_3} R^{c_1\dots c_3}}{3!} + \frac{A_{c_1\dots c_6} R^{c_1\dots c_6}}{6!}\right) \in G_{11}/H_{11} \tag{2}$$

[1] Recall that an algebra is hyperbolic if upon the deletion of any node of the Dynkin diagram the remaining algebra is a direct sum of simple or affine Lie algebras.

[2] For the process of very-extension see also [18, 19, 20].

[3] In order to determine which fields are auxiliary one would require a (still missing) full dynamical understanding of the theory.

The fields $h^a{}_b$, $A_{c_1\ldots c_3}$ and $A_{c_1\ldots c_6}$ depend on the space-time coordinate x^μ. Notice that the non-linear sigma model G_{11}/H_{11} does not produce the bosonic equations of motion of $D = 11$ supergravity. The Cartan forms $\partial_\mu v v^{-1}$ constructed from this coset element for instance do not give rise to the antisymmetrized field strengths. Only after the non-linear realization with respect to the coset of the conformal group with the *same* Lorentz group is also taken does one obtain the correct field strengths and curvature terms of the supergravity theory [21]. Then it is natural to write down the equations of motion in terms of these fields, which then are the full non-linear field equations. Similar calculations were done for the type IIA, IIB and massive IIA supergravity [21, 23, 24], and for the closed bosonic string [25].

- It has been conjectured that an extension of this theory has the rank eleven Kac–Moody algebra called E_{11} as a symmetry [3]. E_{11} is the very-extension of E_8 and is therefore also sometimes called E_8^{+++}. This extension is suggested because, if one drops the momentum generators P_a, the coset G_{11}/H_{11} is a truncation of the coset $E_{11}/K(E_{11})$, as we will see below. Note that dropping the translation generators P_a in a sense corresponds to forgetting about space-time since the field conjugate to P_a are the space-time coordinates x^a. The translation operators P_a will be re-introduced in section 4 where we will also discuss the rôle of space-time in more detail.
- The algebraic structure encoded in E_{11} has been shown to control transformations of Kasner solutions [19] and intersection rules of extremal brane solutions [26]. Furthermore, the relation between brane tensions in IIA and IIB string theory (including the D8 and D9 brane) can be deduced [27, 28].

As mentioned above, similar constructions can be done for all very-extended G^{+++}. In the sequel, the attention is focussed on these models.

The paper is organized as follow. Chapter 2 is devoted to studying the field content of the Kac–Moody models $\mathcal{V}_G$. A brief introduction to Kac–Moody algebras is given in section 2.1. A level decomposition of the generators of a Kac-Moody algebra with respect to a $\mathfrak{gl}(D)$ subalgebra is explained in section 2.2. Such decompositions are of interest since the $\mathfrak{gl}(D)$ subalgebra represents the gravitational degrees of freedom; the other generators are in tensorial representations of $\mathfrak{gl}(D)$ and mostly provide higher spin fields. In section 2.3, the field content of the model $\mathcal{V}_G$ is given for all G^{+++} and the (super-)gravity fields of $\mathcal{O}_G$ are identified. Another remarkable feature of the algebras is how their Dynkin diagrams support different interpretations, in particular T-dualities. This is exemplified in section 2.4. In chapter 3, we offer a few remarks on the relation of the KM fields to higher spin theories. The emergence of space-time in the context of very-extended algebras is envisaged in chapter 4, where we also explain some diagrammatic tricks. The conclusions are given in the last chapter.

2 Field content of Kac–Moody algebras

2.1 Brief definition of Kac–Moody algebra

Let us define a Kac–Moody algebra $\mathfrak{g}$ via its Dynkin diagram with n nodes and links between these nodes. The algebra is a Lie algebra with Chevalley generators $h_i,\ e_i,\ f_i$ $(i = 1, \ldots, n)$ obeying the following relations [29]

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= A_{ij} e_j \\ [h_i, f_j] &= -A_{ij} f_j \\ [h_i, h_j] &= 0 \end{aligned}$$

where $A_{ii} = 2$ and $-A_{ij}$ $(i \neq j)$ is a non-negative integer related to the number of links between the i^{th} and j^{th} nodes. The so-called Cartan matrix A in addition satisfies $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$. The generators must also obey the Serre relations,

$$
\begin{aligned}
(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j &= 0 \\
(\mathrm{ad}\, f_i)^{1-A_{ij}} f_j &= 0
\end{aligned}
$$

A root α of the algebra is a non-zero linear form on the Cartan subalgebra $\mathfrak{h}$ (= the subalgebra generated by the $\{h_i \mid i = 1, ..., n\}$)[4] such that

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$$

is not empty. $\mathfrak{g}$ can be decomposed in the following triangular form,

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

or according to the root spaces,

$$\mathfrak{g} = \oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha \oplus \mathfrak{h}$$

where $\mathfrak{n}_-$ is the direct sum of the negative roots spaces, $\mathfrak{n}_+$ of the positive ones and $\mathfrak{h}$ is the Cartan subalgebra. The dimension of $\mathfrak{g}_\alpha$ is called the multiplicity of α. These multiplicities obey the Weyl–Kac character formula

$$\Pi_{\alpha\in\Delta_+}(1 - e^\alpha)^{\mathrm{mult}\alpha} = \sum_{w\in W} \epsilon(w) e^{w(\rho)-\rho}$$

The sum is over the Weyl group which in the case of interest here is infinite. $\epsilon(w)$ is the parity of w and ρ is the Weyl vector. This formula cannot be solved in closed form in general. We will normally denote by G the (simply-connected, formal) group associated to $\mathfrak{g}$.

Our interest is focussed here on a class of Kac–Moody algebras called *very-extensions* of simple Lie algebras. They are the natural extension of the over-extended algebras which are themselves extensions of the affine algebras. The procedure for constructing an affine Lie algebra $\mathfrak{g}^+$ from a simple one $\mathfrak{g}$ consist in the addition of a node to the Dynkin diagram in a certain way which is related to the properties of the highest root of $\mathfrak{g}$. One may also further increase by one the rank of the algebra $\mathfrak{g}^+$ by adding to the Dynkin diagram a further node that is attached to the affine node by a single line. The resulting algebra $\mathfrak{g}^{++}$ is called the over-extension of $\mathfrak{g}$. The very-extension, denoted $\mathfrak{g}^{+++}$, is found by adding yet another node to the Dynkin diagram that is attached to the over-extended node by one line [18].

A further important concept in this context is that of the *compact form* $K(\mathfrak{g})$ of $\mathfrak{g}$ [29], which we here define as the fixed point set under the compact involution ω mapping

$$\omega(e_i) = -f_i, \qquad \omega(f_i) = -e_i, \qquad \omega(h_i) = -h_i.$$

An element in the coset space of the formal groups $G/K(G)$ then can be parametrized as

$$v = \exp(\sum_i \phi_i h_i) \exp(\sum_{\alpha>0} A_\alpha E_\alpha), \tag{3}$$

where the (infinitely many) positive step operators E_α can have multiplicities greater than one for imaginary roots α.

2.2 Decomposition of a KM algebra under the action of a regular subalgebra

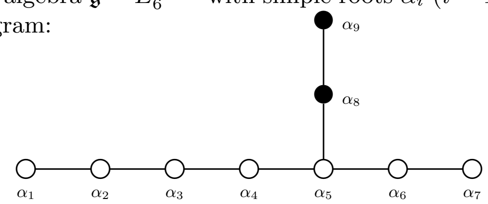
In order to construct a non-linear sigma model associated with a Kac–Moody algebra, e.g. E_8^{+++}, one needs to consider infinitely many step operators E_α and therefore infinitely many corresponding fields A_α because the algebra is infinite-dimensional.[5] To date, only truncations of this model can be constructed.

[4]Here we assume the Cartan matrix A to be non-degenerate.

[5]See also appendix A for the notation.

A convenient organization of the data is to decompose the set of generators of a given Kac–Moody algebra $\mathfrak{g}$ with respect to a finite regular subalgebra $\mathfrak{s}$ and a corresponding *level decomposition*.[6] The 'level' provides a gradation on $\mathfrak{g}$ and the assignment is as follows: generators in $\mathfrak{h}$ are at level $\ell = 0$ and elements in $\mathfrak{g}_\alpha$ derive their level from the root α, usually as a subset of the root labels. The following example is taken to illustrate the notion of level.

Level of a root: Consider the algebra $\mathfrak{g} = E_6^{+++}$ with simple roots α_i $(i = 1, \dots, 9)$ labelled according to the Dynkin diagram:



We want to make a level decomposition of E_6^{+++} under its $\mathfrak{s} = A_7$ subalgebra corresponding to the sub-Dynkin diagram with nodes from 1 to 7. This singles out the nodes 8 and 9 which do not belong to the A_7 subdiagram. Any positive root α of E_6^{+++} can be written as $\alpha = \sum_{i=1}^{9} m_i\alpha_i = \sum_{s=1}^{7} m_s\alpha_s + \sum_{g=8,9} \ell_g\alpha_g$ with m_s and ℓ_g non-negative integers. Here, ℓ_8 and ℓ_9 are called respectively the α_8 level and the α_9 level of α.

The truncation consists of considering only the generators of the lowest levels; the coset element (3) is then calculated by using only the fields corresponding to these generators. The question of which cut-off to take for the truncation will be answered universally below.

The adjoint action of $\mathfrak{s}$ on $\mathfrak{g}$ preserves the level, therefore the space of generators on a given level is a (finite-dimensional) representation space for a representation of $\mathfrak{s}$ and hence completely reducible. The fields A_α associated with roots of a given level $\{\ell_g\}$ will be written as representations of $\mathfrak{s}$.

In order to obtain an interpretation of the fields associated with these generators as space-time fields, it is convenient to choose the regular subalgebra to be $\mathfrak{s} = \mathfrak{sl}(D)$ for some D; these are always enhanced to $\mathfrak{gl}(D)$ by Cartan subalgebra generators associated with the nodes which were singled out. The reason for the choice of an A-type subalgebra is that one then knows how the resulting tensors transform under $\mathfrak{so}(D) = K(\mathfrak{s}) \subset K(\mathfrak{g})$. In the spirit of the non-linear realization explained in the introduction, this $\mathfrak{so}(D)$ plays the rôle of the local Lorentz group.[7] In particular, the level $\ell = 0$ sector will give rise to the coset space $GL(D)/SO(D)$ since it contains the adjoint of the regular $\mathfrak{gl}(D)$ subalgebra. This is the right coset for the gravitational vielbein which is an invertible matrix defined up to a Lorentz transformation.[8] If the rank of $\mathfrak{g}$ is r then there will also be $r - D$ additional dilatonic scalars at level $\ell = 0$ from the remaining Cartan subalgebra generators.

Let us study in more detail which representations of $\mathfrak{s}$ can occur at a given level. The necessary condition for a positive root $\alpha = \sum_s m_s\alpha_s + \sum_g \ell_g\alpha_g$ of $\mathfrak{g}$ to generate a lowest weight representation of $\mathfrak{s}$ are that

$$p_s = -\sum_t A_{st}m_t - \sum_g A_{sg}\ell_g \geq 0$$

where A is the Cartan matrix of $\mathfrak{g}$, the $\{p_s\}$ are the Dynkin labels of the corresponding lowest weight representation.[9] We can also rephrase this condition by realizing that a representation with labels $\{p_s\}$

[6]We restrict to finite-dimensional subalgebras $\mathfrak{s}$ in order to have only a finite number of elements for each level [5,30].

[7]The real form $\mathfrak{so}(D)$ is actually not the correct one for an interpretation as Lorentz group. In order to obtain the correct $\mathfrak{so}(D-1,1)$ one needs to modify the compact involution to a so-called 'temporal involution' [6]. As discussed in [31,32] this leads to an ambiguity in the signature of space-time since Weyl-equivalent choices of involution result in inequivalent space-time signatures on the 'compact' part of the subalgebra $\mathfrak{s}$. We will not be concerned with this important subtlety here as it does not affect the higher spin field content.

[8]For a recent 'stringy' decomposition of E_{10} under a subalgebra of type D see [33].

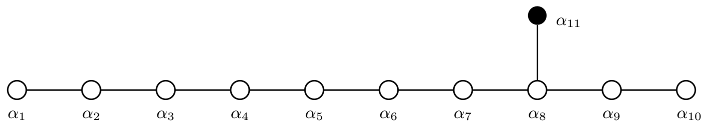
[9]Actually the lowest weight then has Dynkin labels $\{-p_s\}$ since we have introduced an additional minus sign in the conversion between the different labels which is convenient for the class of algebras we are considering here.

at level $\{\ell_g\}$ corresponds to a root with coefficients

$$m_s = -\sum_t (A_{\rm sub}^{-1})_{st} p_t \sum_{t,g} -(A_{\rm sub}^{-1})_{st} A_{tg} l_g \geq 0$$

and these are non-negative integers if they are to belong to a root of $\mathfrak{g}$. Here, $A_{\rm sub}$ is the Cartan matrix of $\mathfrak{s}$. Besides this necessary condition one has to check that there are elements in the root space of α that can serve as highest weight vectors. This requires calculating the multiplicity of α as a root of $\mathfrak{g}$ and its weight multiplicity in other (lower) representations on the same level. The number of independent highest weight vectors is called the *outer multiplicity* of the representation with labels $\{p_s\}$ and usually denoted by μ.

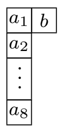
Example: $\mathfrak{g} = E_8^{+++}$, which is the very extension of E_8, can be decomposed w.r.t its subalgebra $\mathfrak{s} = A_{10} = \mathfrak{sl}(11)$ associated with the sub-Dynkin diagram with nodes from 1 to 10,



One finds that the level $\ell \equiv \ell_{11} = 0$, 1, 2 and 3 generators are in a lowest weight representation of A_{11} with Dynkin labels given in the following table,

ℓ	Dynkin labels	Tensor
0	[1, 0, 0, 0, 0, 0, 0, 0, 0, 1] + [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]	$K_a{}^b$
1	[0, 0, 0, 0, 0, 0, 0, 1, 0, 0]	$R^{a_1a_2a_3}$
2	[0, 0, 0, 0, 1, 0, 0, 0, 0, 0]	$R^{a_1a_2a_3a_4a_5a_6}$
3	[0, 0, 1, 0, 0, 0, 0, 0, 0, 1]	$R^{a_1\dots a_8,b}$
...		

The level zero generators are associated with the gravitational degrees of freedom as explained above. The level one field can be recognized as the three form potential of the supergravity in $D = 11$ and the level two field correspond to its dual. Notice that up to level 2 the generators are just the fields of G_{11}, mentioned in the introduction, except for the momentum generators P_a. The level three tensor corresponds to a Young tableau of the form



and is associated with the dual of the graviton [34, 35, 36].[10] The dualization of Einstein's field equations only works at the linearized level however [36]. One might speculate that a dualization of the non-linear equations will probably require some of the remaining fields in the infinite list of tensors above level 3. In summary, the lowest level generators ($\ell = 0, 1, 2, 3$) of the E_8^{+++} algebra correspond to the degrees of freedom (and their duals) of $D = 11$ supergravity, which is $\mathcal{O}_{E_8^{+++}}$.

[10] There is a subtlety here since the $\mathfrak{gl}(11)$ tensor associated with this mixed symmetry vanishes after antisymmetrization over all indices. This means that one cannot accomodate the trace of the spin connection in this dual picture.

$\mathfrak{g}^{+++}$	oxidized theory (maximal)
E_8^{+++}	$N=1$, $D=11$ SUGRA
E_7^{+++}	$N=IIB$, $D=10$ SUGRA, truncated non-supersymmetrically to dilaton and 4-form potential with self-dual field strength
E_6^{+++}	$N=2$, $D=8$ SUGRA, truncated to dilaton, axion and 3-form potential
D_n^{+++}	Massless sector of the closed bosonic string in $D=n+2$
A_n^{+++}	Gravity in $D=n+3$
G_2^{+++}	Einstein-Maxwell $N=2$, $D=5$ SUGRA
F_4^{+++}	$N=(0,1)$, $D=6$ chiral SUGRA
B_n^{+++}	massless 'heterotic' string in $D=n+2\equiv$ closed bosonic string coupled to a massless abelian vector potential
C_n^{+++}	$D=4$ theory, $(n-1)^2$ scalars and $2(n-1)$ vector potentials transforming under C_n

Table 4.1: The list of oxidized theories as recovered by very-extended Kac–Moody algebras $\mathfrak{g}^{+++}$.

2.3 Results

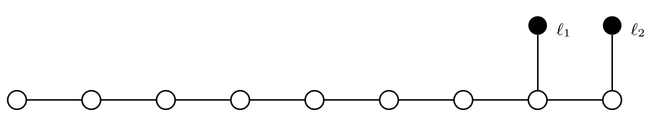
The analysis above can be repeated for all very-extended algebras $\mathfrak{g}^{+++}$. There is a natural maximal choice for the $\mathfrak{sl}(D)$ algebra [19,20]. This is obtained by starting at the very-extended node and following the line of long roots as far as possible.

The decomposition under this maximal gravity subalgebra, truncated at the level of the affine root of $\mathfrak{g}^{+}$, corresponds precisely to the bosonic fields of the oxidized theory $\mathcal{O}_G$, if one also includes *all* dual fields for form matter and gravity [20]. Instead of repeating the full analysis we summarize the fields by their oxidized theories in table 4.1; the Kac–Moody algebra $\mathfrak{g}^{+++}$ captures only the bosonic fields. The truncation criterion of the affine root seems natural since one knows that the finite-dimensional algebras $\mathfrak{g}$ correspond to the oxidized theories and generators in the 'true Kac–Moody sector' are likely to have a different rôle.

All oxidized theories are theories containing gravity and pure gravity itself is associated with A type algebras. It can be shown that on the level of very-extended algebras the relevant maximal A_{D-3}^{+++} algebra is contained in the M-theory algebra $\mathfrak{g}^{+++}$ [20]. .

2.4 Different embeddings

We can analyse the field content of a theory with respect to different embeddings, i.e. different choices of A_n subalgebras. For example, the type IIA supergravity theory is associated with the following decomposition of E_8^{+++} (we now only mark in black the nodes to which we assign a level ℓ_i),

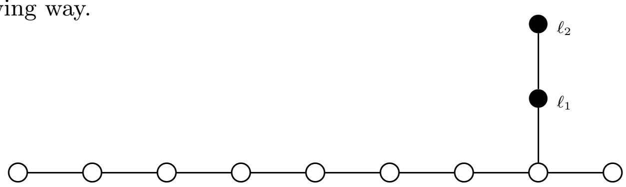


The level (ℓ_1,ℓ_2) generators of E_8^{+++} corresponding to the roots of the form $\alpha=\sum_s m_s\alpha_s+\ell_1\alpha_{\ell_1}+\ell_2\alpha_{\ell_2}$ are in representations of A_9 characterized by the Dynkin labels $\{p_s\}=[p_1,...,p_9]$. The following table gives, up to the level of the affine root, the Dynkin labels of the representations occurring in E_8^{+++} and their outer multiplicities μ, as well as the interpretation of the fields.

(ℓ_1,ℓ_2)	$[p_1,...,p_9]$	μ	Interpretation
(0,0)	[1,0,0,0,0,0,0,0,1]$\oplus$[0,0,0,0,0,0,0,0,0]	1	$h_a{}^b$
(0,0)	[0,0,0,0,0,0,0,0,0]	2	ϕ
(1,0)	[0,0,0,0,0,0,0,1,0]	1	$B_{(2)}$
(0,1)	[0,0,0,0,0,0,0,0,1]	1	$A_{(1)}$
(1,1)	[0,0,0,0,0,0,1,0,0]	1	$A_{(3)}$
(2,1)	[0,0,0,0,1,0,0,0,0]	1	$\tilde{A}_{(5)}$
(2,2)	[0,0,0,1,0,0,0,0,0]	1	$\tilde{B}_{(6)}$
(3,1)	[0,0,1,0,0,0,0,0,0]	1	$\tilde{A}_{(7)}$
(3,2)	[0,0,1,0,0,0,0,0,1]	1	$\tilde{A}_{(7,1)}$
(3,2)	[0,1,0,0,0,0,0,0,0]	1	$\tilde{A}^{\phi}_{(8)}$
...			

These fields match the field content of type IIA supergravity. Indeed, in addition to the level zero gravitational fields and their dual fields, one recognizes the NS-NS two form $B_{(2)}$, the RR one form $A_{(1)}$, the RR three form $A_{(3)}$ and the dual $\tilde{B}_{(6)}$, $\tilde{A}_{(7)}$ and $\tilde{A}_{(5)}$, respectively.

For type IIB supergravity, the choice of the A_9 subalgebra can be depicted by redrawing the E_8^{+++} Dynkin diagram in the following way.



The decomposition produces the following table [20].

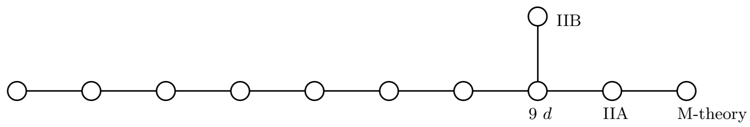
(ℓ_1,ℓ_2)	$[p_1,...,p_9]$	μ	Interpretation
(0,0)	[1,0,0,0,0,0,0,0,1]$\oplus$[0,0,0,0,0,0,0,0,0]	1	$h_a{}^b$
(0,0)	[0,0,0,0,0,0,0,0,0]	2	ϕ
(1,0)	[0,0,0,0,0,0,0,0,0]	1	χ
(0,1)	[0,0,0,0,0,0,0,1,0]	1	$B_{(2)}$
(1,1)	[0,0,0,0,0,0,0,1,0]	1	$A_{(2)}$
(1,2)	[0,0,0,0,0,1,0,0,0]	1	$A_{(4)}$
(1,3)	[0,0,0,1,0,0,0,0,0]	1	$\tilde{B}_{(6)}$
(2,3)	[0,0,0,1,0,0,0,0,0]	1	$\tilde{A}_{(6)}$
(2,4)	[0,0,1,0,0,0,0,0,1]	1	$\tilde{A}_{(7,1)}$
(1,4)	[0,1,0,0,0,0,0,0,0]	1	$\tilde{A}^{\chi}_{(8)}$
(2,4)	[0,1,0,0,0,0,0,0,0]	1	$\tilde{A}^{\phi}_{(8)}$
...			

The fields in the table correspond to the gravitational fields $h_a{}^b$ and the dilaton ϕ, the RR zero form (axion) χ, the NS-NS two form $B_{(2)}$, the RR two form $A_{(2)}$ and their duals $\tilde{A}_{(7,1)}$, $\tilde{A}^{\phi}_{(8)}$, $\tilde{A}^{\chi}_{(8)}$, $\tilde{B}_{(6)}$, $\tilde{A}_{(6)}$, respectively. The RR four form $A_{(4)}$ will have self-dual field strength. Remember that the field content corresponds only to the few lowest level generators. The tables continue infinitely but there is no interpretation to date for the additional fields, most of which have mixed Young symmetry type.[11]

[11] We remark that there are two exceptions of fields which have a reasonable possible interpretation [20]. These occur in the IIA and IIB decomposition of E_8^{+++} and correspond to the nine-form potential in massive IIA supergravity and to the ten-form potential in IIB theory, which act as sources of the D8 in and D9 brane in IIA and IIB superstring theory, respectively.

As string theories, type IIA and IIB string theories are related by 'T-duality'. Here we notice a diagrammatic reflection of this fact through the 'T-junction' in the Dynkin diagram of E_8^{+++}. Therefore, the diagrams of the very-extended Kac–Moody algebras nicely encode the field content of the oxidized theories and also relations between different theories.

Moreover, Kaluza–Klein reduction corresponds to moving nodes out of the 'gravity line' (the A-type subalgebra). For example, we saw that the bosonic field content of $D = 11$ supergravity can be retrieved by considering the Dynkin diagram of E_8^{+++} with a $\mathfrak{sl}(11)$ subalgebra so that the endpoint of the gravity line is the 'M-theory' node in the diagram below. A reduction of $D = 11$ supergravity produces IIA supergravity and the endpoint of the corresponding gravity line is also marked in the diagram in agreement with the analysis above. Finally, the ten-dimensional IIB theory is not a reduction of $D = 11$ supergravity but agrees with it after reduction to nine dimensions. Roughly speaking this means taking a different decompactification after $D = 9$ not leading to M-theory. This is precisely the structure of the E_8^{+++} diagram.



3 Relation to higher spin fields

We now turn to the question in what sense the fields contained in the Kac–Moody algebras are true higher spin fields. Our minimal criterion will be that they transform under the *space-time* Lorentz group and that their dynamics is consistent. These requirements will be discussed for the two familiar classes of Kac–Moody models.

1. West's proposal: As explained in the introduction, in [21, 3] the Kac–Moody algebra is a symmetry of the unreduced theory. Therefore, the fields transform under local space-time $SO(D) \subset K(G^{+++})$ transformations and therefore satisfy the first requirement. But as we have also stressed, the derivation of the dynamical equations requires the introduction of additional translation generators and the closure with the conformal algebra to obtain the correct curvatures, at least for gravity and the anti-symmetric matter fields. It is not known what this procedure yields for the mixed symmetry fields at higher levels. When writing down the dynamical equations there is also an ambiguity in numerical coefficients which should ultimately be fixed from the algebraic structure alone, maybe together with supersymmetry. At the present stage the consistency of the higher spin dynamics cannot be determined.

However, it might be that known higher spin formulations play a rôle in the resolution of these difficulties. In particular, the 'unfolded dynamics' of [37,38][12] could provide a dynamical scheme for the KM fields.

2. Null geodesic world-lines on Kac–Moody coset spaces: This is the approach taken in [5, 6], where the idea is to map a one-dimensional world-line, and not space-time itself, into the coset space $G^{+++}/K(G^{+++})$. The advantage of this approach is that one does not require translation operators or the closure with the conformal group. Rather space-time is conjectured to re-emerge from a kind of Taylor expansion via gradient fields of the oxidized fields present in the decomposition tables. For the details see [5] or the discussion in [40]. Therefore space-time is thought of as a Kac–Moody intrinsic concept in the world-line approach. The dynamical equations in this context are derived from a lagrangian formulation for the particle motion.

¿From the higher spin point-of-view the fields no longer are true higher spin fields under the space-time Lorentz group but rather under some internal Lorentz group. In order to transform the KM fields into honest higher spin fields the recovery of space-time from the KM algebra needs to be made precise. On the other hand, the dynamical scheme here is already consistent and therefore our second requirement for higher spin dynamics is fulfilled automatically.

[12] See also M. Vasiliev's contribution to the proceedings of this workshop.

4 Space-time concepts

Minkowski space-time can be seen as the quotient of the Poincaré group by the Lorentz group. The Poincaré group itself is the semi-direct product of the Lorentz group with the (abelian) group of translations and the translations form a vector representation of the Lorentz group:

$$[M^{ab}, M^{cd}] = \eta^{ac}M^{bd} - \eta^{ad}M^{bc} + \eta^{bd}M^{ac} - \eta^{bc}M^{ad}, \tag{4}$$
$$[P_a, P_b] = 0, \tag{5}$$
$$[M^{ab}, P_c] = \delta^a_c \eta^{bd} P_d - \delta^b_c \eta^{ad} P_d. \tag{6}$$

In the Kac-Moody context, the Lorentz group is replaced by $K(G^{+++})$ so one needs a 'vector representation' of $K(G^{+++})$. By this we mean a representation graded as a vector space by level $\ell \geq 0$ with a Lorentz vector as bottom component at $\ell = 0$, corresponding to the vector space decomposition of the compact subalgebra

$$K(\mathfrak{g}^{+++}) = \mathfrak{so}(D) \oplus ...$$

Unfortunately, very little is known about $K(\mathfrak{g}^{+++})$, except that it is not a Kac–Moody algebra. However, it is evidently a subalgebra of $\mathfrak{g}^{+++}$. One can therefore construct representations of $K(\mathfrak{g}^{+++})$ by taking a representation of $\mathfrak{g}^{+++}$ and then view it as a $K(\mathfrak{g}^{+++})$ module. Irreducibility (or complete reducibility) of such representations are interesting open questions. If we take a (unitary) lowest weight representation of $\mathfrak{g}^{+++}$ then we can write it as a tower of $\mathfrak{gl}(D)$-modules in a fashion analogous to the decomposition of the adjoint. The idea in reference [39] was to take a representation of $\mathfrak{g}^{+++}$ whose bottom component is a $\mathfrak{gl}(D)$ vector for any choice of gravity subalgebra. This is naturally provided for by the $\mathfrak{g}^{+++}$ representation with lowest weight Dynkin labels

$$[1, 0, ..., 0]_{\mathfrak{g}^{+++}}.$$

We denote this irreducible $\mathfrak{g}^{+++}$ representation by $L(\lambda_1)$ since the lowest weight is just the fundamental weight of the first (very-extended) node.[13] Decomposed w.r.t. the gravity subalgebra A_{D-1}, this yields

$$[\underbrace{1, 0, \ldots, 0}_{r \text{ labels}}]_{\mathfrak{g}^{+++}} \quad \to \quad [\underbrace{1, 0, \ldots, 0}_{(D-1) \text{ labels}}]_{gravity} \oplus \ldots$$

where the "..." denote the infinitely many other representations of the gravity subalgebra at higher levels (r is the rank of $\mathfrak{g}^{+++}$). These fields can be computed in principle with the help of a character formula. For example, for E_8^{+++}, one finds the following decomposition of the vectorial representation w.r.t. the M-theory gravity subalgebra A_{10} [39]:

ℓ	$[p_1, \ldots, p_{10}]$	Field
0	$[1,0,0,0,0,0,0,0,0,0]$	P_a
1	$[0,0,0,0,0,0,0,0,1,0]$	Z^{ab}
2	$[0,0,0,0,0,1,0,0,0,0]$	$Z^{a_1\ldots a_5}$
3	$[0,0,0,1,0,0,0,0,0,1]$	$Z^{a_1\ldots a_7,b}$
3	$[0,1,0,0,0,0,0,0,0,0]$	$Z^{a_1\ldots a_8}$
	...	

In addition to the desired vector representation P_a of $\mathfrak{gl}(11)$, one finds a 2-form and a 5-form at levels $\ell = 1, 2$. It was noted in [39] that these are related to the $D = 11$ superalgebra

$$\{Q_\alpha, Q_\beta\} = \big((\Gamma^a C)P_a + \frac{1}{2}(\Gamma_{ab}C)Z^{ab} + \frac{1}{5!}(\Gamma_{a_1\ldots a_5}C)Z^{a_1\ldots a_5}\big)_{\alpha\beta} \tag{7}$$

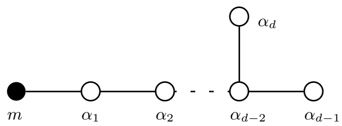
in an obvious way. (C is the charge conjugation matrix.) The branes of M-theory couple as $\frac{1}{2}$-BPS states to these central charges. Therefore the vector representation also encodes information about the

[13] Therefore, the representation is integrable and unitarizable.

topological charges of the oxidized theory. We also note that the charges on $\ell = 3$ suggest that they might be carried by solitonic solutions associated with the dual graviton field, but no such solutions are known.[14] Therefore, we will refer to the tensors in the $\mathfrak{g}^{+++}$ representation $L(\lambda_1)$ as 'generalized central charges' [40].[15]

The result carries over to all algebras $\mathfrak{g}^{+++}$: To any field we found in the decomposition for the oxidized theory there is an associated generalized charge [40] in the 'momentum representation" $L(\lambda_1)$. For instance, for an anti-symmetric p-form this is just a $(p-1)$-form. It is remarkable that this result holds for all $\mathfrak{g}^{+++}$ and not only for those where one has a supersymmetric extension of the oxidized theory.

To demonstrate this result we use a trick [40] which turns the statement into a simple corollary of the field content analysis. Instead of using the Weyl–Kac or the Freudenthal character formulae, we embed the semi-direct sum of the algebra with its vectorial representation in a larger algebra [40]. This can be illustrated nicely for the Poincaré algebra. The aim is to find a minimal algebra which contains (as a truncation) both the original Lorentz algebra and the vector representation by Dynkin diagram extensions. We consider the case of even space-time dimension for simplicity. The $\mathfrak{so}(2d)$ vector has Dynkin labels $[1, 0, \ldots, 0]$ in the standard choice of labelling the simple nodes. Hence, adding a node (marked in black in the diagram below, with label m for momentum) at the position 1 below with a single line will give a vector in the decomposition under the Lorentz algebra at $\ell = 1$.



The resulting embedding algebra is the conformal algebra as the full decomposition shows[16]

$$\{M^{ab}, P^c\} \to_{embed} M^{AB} \to_{decompose} \{M^{ab}, P^c, K^c, D\}, \tag{8}$$

where M^{ab} $(a = 1, ..., 2d)$, P^a, K^a and D are respectively the generators of the Lorentz group, the translations generators, the special conformal transformations generators and the dilatation operator in D dimensions. M^{AB} $(A = 1, ..., 2d+2)$ are the generators of the conformal algebra $\mathfrak{so}(2d+1, 1)$. (Note that if one allows a change in the original diagram then there is smaller embedding in the AdS algebra $\mathfrak{so}(2d, 1)$ [41]; a fact that has also been exploited in the higher spin literature.)

Returning to the KM case, one embeds the semi-direct product of $\mathfrak{g}^{+++}$ with its vector representation in a larger algebra in a similar fashion. The resulting algebra is obtained by adding one more node to the Dynkin diagram of $\mathfrak{g}^{+++}$ at the very-extended node and we denote this fourth extension of $\mathfrak{g}$ by $\mathfrak{g}^{++++}$:

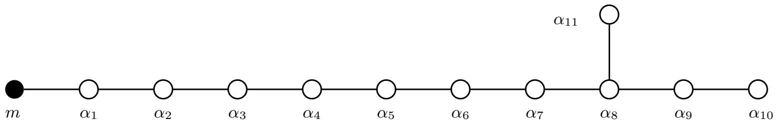
$$\mathfrak{g}^{+++} \ltimes L(\lambda_1) \hookrightarrow \mathfrak{g}^{++++}. \tag{9}$$

Note that the embedding is only valid at the level of vector spaces; the Lie algebra relations in the two spaces are different, since not all the central charges will commute any longer in $\mathfrak{g}^{++++}$. (However, the momenta P_a will commute.) The Dynkin diagram of $\mathfrak{g}^{++++}$, for the very-extension E_8^{+++}, is the following:

[14] This idea has appeared before in the context of U-duality where the mixed symmetry 'charge' was associated with a Taub-NUT solution [42].

[15] These are generalized since it is not clear in what algebra they should be central and the name is just by analogy. Note, however, that these charges will still all commute among themselves.

[16] The diagram only shows this for the complex algebras but one can actually check it at the level of real forms as well.



By using this kind of embedding, it follows that one obtains all generalized central charges for the KM fields of the oxidized theories. This is a consequence of a Kaluza–Klein reduction of the oxidized fields: A field in D space-time dimensions gets lifted to one in $D+1$ dimensions by the diagram extension and subsequent reduction then results in the field in D dimensions and its corresponding charge. We remark that in this construction of space-time there are infinitely many space-time generators, rendering space-time infinite-dimensional. (The idea of associating space-time directions to central charges goes back to [43].)

5 Conclusions

We have demonstrated that indefinite Kac–Moody algebras are a rich natural source for higher spin fields with mixed symmetry type. In all existing proposals, however, there are important open problems in deriving true and consistent higher spin field theories from this algebraic approach to M-theory. As we indicated, known results from higher spin theory might be used to solve some of these problems. Further progress might be derived from an understanding of the KM structure to all levels which appears to be a very hard problem.

All the models we discussed so far are for bosonic fields, an extension to fermionic degrees of freedom relies on an understanding of the compact subalgebra $K(\mathfrak{g})$. The next step then would be to relate these to the fermionic (or supersymmetric) HS theories *à la* Fang–Fronsdal.

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A Coset Models

Let us recall some basic facts about non-linear sigma models associated with the coset model $G/K(G)$. We use the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G,

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$$

where $\mathfrak{k}$ is the Lie algebra corresponding to $K(G)$ and is fixed by an involution ω of $\mathfrak{g}$:

$$\mathfrak{k} = \{x \in \mathfrak{g} \mid \omega(x) = x\},$$
$$\mathfrak{p} = \{x \in \mathfrak{g} \mid \omega(x) = -x\}.$$

We will here take ω to be the compact involution, i.e. a map from $\mathfrak{g}$ to $\mathfrak{g}$ such that

$$\omega(e_i) = -f_i \; ; \; \omega(f_i) = -e_i \; ; \; \omega(h_i) = -h_i$$

on the r Chevalley generators of $\mathfrak{g}$. Let us consider the coset element v in Iwasawa parametrization

$$v(x) = \exp(\overrightarrow{\phi}(x).\overrightarrow{h})\exp\,(\sum_{\alpha\in\Delta_+} A_\alpha(x)E^\alpha) \in G/K(G)$$

where the $\overrightarrow{h}$ are the generators of the Cartan subalgebra and the E^α's are the step operators associated with the positive roots. The coordinates on $G/K(G)$ namely $\overrightarrow{\phi}$ and A_α depend on the space-time coordinates. dvv^{-1} is in $\mathfrak{g}$, and hence we can decompose it as $dvv^{-1} = \mathcal{P} + \mathcal{Q} = (\mathcal{P}_\mu + \mathcal{Q}_\mu)dx^\mu$. A natural lagrangian associated to this model is

$$L = \frac{1}{4n}(\mathcal{P}_\mu|\mathcal{P}^\mu)$$

where the inner product $(\cdot|\cdot)$ is the invariant metric on $\mathfrak{g}$. Such a lagrangian is invariant under local $K(G)$ transformations and global G transformations,

$$v(x) \;\rightarrow\; k(x)v(x)g$$

where $k(x)$ is in $K(G)$ and g is in G. The lagrangian equations of motion are

$$D^\mu(n^{-1}\mathcal{P}_\mu) = (\partial^\mu - \mathrm{ad}\,\mathcal{Q}^\mu)(n^{-1}\mathcal{P}_\mu) = 0 \tag{A.1}$$

Furthermore, the Lagrange multiplier n constrains the motion to be null, which is covariantly constant along the coset by (A.1). An introduction to scalar coset models can be found in [44].

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Holography, Duality and Higher-Spin Theories

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Abstract. I review recent work on the holographic relation between higher-spin theories in Anti-de Sitter spaces and conformal field theories. I present the main results of studies concerning the higher-spin holographic dual of the three-dimensional $O(N)$ vector model. I discuss the special role played by certain double-trace deformations in Conformal Field Theories that have higher-spin holographic duals. Using the canonical formulation I show that duality transformations in a $U(1)$ gauge theory on AdS_4 induce boundary double-trace deformations. I argue that a similar effect takes place in the holography linearized higher-spin theories on AdS_4.

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1 Introduction

Consistently interacting higher-spin (HS) gauge theories exist on Anti-de Sitter spaces (see [1, 2] for recent reviews and extensive literature on higher-spin gauge theories). It is then natural to ask whether HS theories have interesting holographic duals. In this lecture I review recent work on the holographic relation between HS theories and conformal field theories (CFTs). After some general remarks concerning the relevance of HS theories to free CFTs, I present the main results in studies of the three-dimensional critical $O(N)$ vector model that has been suggested to realize the holographic dual of a HS theory on AdS$_4$. Furthermore, I discuss the special role the so-called double-trace deformations seem to play in the dynamics of CFTs that have holographic HS duals. I particular, I show that duality transformations of a $U(1)$ gauge theory on AdS$_4$ induce boundary double-trace deformations. I argue that a similar effect takes place in the implementation of holography to linearized HS theories on AdS$_4$.

2 Higher-spin currents and the operator spectrum of Conformal Field Theories

The operator spectrum of d-dimensional Conformal Field Theories (CFTs) consists of an infinite set of modules each containing one quasi-primary operator $\Phi(0)$ that is annihilated by the generator $\hat{K}_\mu$ of special conformal transformations [3]

$$\hat{K}_\mu \Phi(0) = 0 \, , \tag{1}$$

as well as an infinite number of descendants that are essentially the derivatives of $\Phi(0)$. Thus, quasi-primary operators carry irreps of the conformal group $SO(d,2)$ labeled only by their spin s and scaling dimension Δ. As a consequence, the 2- and 3-pt functions of quasi-primary operators are determined up to a number of constant parameters (see for example [4]).

Perhaps the most important property of quasi-primary operators is that they form an algebra under the (associative) operator product expansion (OPE) [5]. Given two such operators $A(x)$ and $B(x)$ we may expand their product in a (infinite) set of quasi-primary operators $\{Q\}$ as

$$A(x)B(0) = \sum_{\{Q\}} C(x,\partial)Q(0) \, . \tag{2}$$

The coefficients $C(x,\partial)$ are fully determined in terms of the spins, dimensions and 3-pt functions of the operators involved in the OPE. Knowledge of the OPE is potentially sufficient to determine all correlation functions of quasi-primary operators in CFTs.

In specific models all quasi-primary operators (with the exception of the "singletons"), are composite operators and determining their precise list, spins and dimensions is a hard task which is done in practice by studying 4-pt functions. For example, consider a scalar quasi-primary operator $\Phi(x)$ with dimension Δ. The OPE enables us to write

$$\begin{aligned}
\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle &= \sum_{\{Q\}} C(x_{12},\partial_2)C(x_{34},\partial_4)\langle Q(x_2)Q(x_4)\rangle \\
&= = \frac{1}{(x_{12}^2 x_{34}^2)^\Delta} \sum_{\{\Delta_s,s\}} g_{\Delta_s,s}\, H_{\Delta_s,s}(v,Y) \\
&= \frac{1}{(x_{12}^2 x_{34}^2)^\Delta} \sum_{\{\Delta_s,s\}} g_{\Delta_s,s}\, v^{\frac{\Delta_s - s}{2}} Y^s \Big[1 + O(v,Y)\Big] \, ,
\end{aligned} \tag{3}$$

where we have used the standard harmonic ratios

$$v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \, , \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2} \, , \qquad Y = 1 - \frac{v}{u}, \qquad x_{ij}^2 = |x_i - x_j|^2 \, . \tag{4}$$

The coefficients $g_{\Delta_s,s}$ are the "couplings". The expressions $H_{\Delta_s,s}(u,Y)$ are complicated but explicitly known [6] functions that give the contribution of an operator with spin s and dimension Δ to the OPE. The terms $v^{(\Delta_s-s)/2}Y^s$ correspond to the leading short-distance behavior of the 4-pt function for $x_{12}^2\,, x_{34}^2 \to 0$.

In the simplest case of a massless free CFT the set $\{Q\}$ includes, among others, all symmetric traceless quasi-primary operators with spin s and dimension Δ_s. It is a general result in CFT that if the above operators are also conserved, then their dimensions are given by [1]

$$\Delta_s = d - 2 + s\,. \tag{5}$$

Therefore, whenever higher-spin conserved currents reside in the list of quasi-primary operators each one contributes in (3) a term of the form

$$v^{\frac{d-2}{2}}Y^s\,. \tag{6}$$

Of course, irreps with canonical scaling dimension (5) are nothing but the massless UIRs of the $d+1$-dimensional Anti-de Sitter group.

Suppose now that one is interested in the holographic description of the 4-pt function (3). For that, we notice that from a classical action on Anti-de Sitter we get non-trivial boundary correlators by studying tree-level bulk-to-boundary graphs [8]. Hence, when the boundary 4-pt function contains terms like (6) it is necessary to consider bulk massless currents. On the other hand, the energy-momentum tensor always appears in the OPE of a 4-pt function, it has dimension d and spin-2. Its dimension remains canonical. Its contribution to the 4-pt function is given by the (rather complicated) function

$$\begin{aligned} H_{e.m.}(v,Y) &= vF_1(Y)[1+O(v,Y)]\,, && (7)\\ F_1(Y) &= \frac{4Y^2-8Y}{Y^3} + \frac{4(-6+6Y-Y^2)}{Y^3}\ln(1-Y) \longrightarrow Y^2 + ... && (8)\end{aligned}$$

A concrete realization of the ideas above is provided by explicit calculations in $\mathcal{N}=4$ SYM$_4$ via the AdS/CFT correspondence [9]. Consider the 4-pt function of the so-called lower dimension chiral primary operators (CPOs) of $\mathcal{N}=4$, which are scalar operators with dimension 2. In the free field theory limit we have

$$\begin{aligned} &\frac{\delta^{I_1I_2}\delta^{I_3I_4}}{400}\langle Q^{I_1}(x_1)...Q^{I_4}(x_4)\rangle_{free} = \\ &\qquad = \frac{1}{(x_{12}^2x_{34}^2)^2}\Big(1+\frac{1}{20}v^2+\frac{1}{20}v^2(1-Y)^{-2} \\ &\qquad + \frac{4}{N^2}(\frac{1}{6}[v+v(1-Y)^{-1}]+\frac{1}{60}v^2(1-Y)^{-1})\Big) \\ &\qquad = \frac{1}{(x_{12}^2x_{34}^2)^2}\Big(...+\frac{4}{6N^2}\sum_{l=2}^{\infty}vY^l+...\Big)\,. \end{aligned} \tag{9}$$

In the last line of (9) we see the contribution of an infinite set of higher-spin conserved currents in the *connected* part of the correlator. The leading contribution comes from the energy-momentum tensor. One can identify the contributions from all the higher-spin currents and even calculate their "couplings", after some hard work that involves the subtraction of the descendants of each and every current. The perturbative corrections to the connected part of the free result above have the form

$$[connected] = \frac{1}{N^2}[connected]_{free} + \frac{1}{N^2}g_{YM}^2N\,F(v,Y)\,, \tag{10}$$

where

$$F(v,Y) \sim \sum_l v\,Y^l\ln v + ... \tag{11}$$

[1]The converse is not always true [7], therefore even if one finds terms such as (6) in the OPE one should be careful in interpreting them.

The above terms can be attributed to an infinite set of "nearly conserved" higher-spin currents i.e. quasi-primary operators whose scaling dimensions have being shifted from their canonical value as [9]

$$\Delta_{HS} \longrightarrow \Delta_{HS} + \gamma = 2 + s + (g^2_{YM}N)\eta_s + \cdots \,. \tag{12}$$

By the operator/state correspondence in CFTs, we may view the above effect as a small deformation of the energy spectrum of the theory. From the AdS side this deformation should correspond to a Higgs-like effect by which the initially massless higher-spin currents acquire masses [10, 11].

In the context of the AdS_5/CFT_4 correspondence, one can calculate the same 4-pt function using IIB supergravity [12]. The result is highly non-trivial and looks like

$$[connected] = \frac{1}{N^2}\frac{1}{(x^2_{12}x^2_{34})^2}\Big[vF_1(Y) + O(v^2, Y)\Big] \,. \tag{13}$$

It is an astonishing fact that the expansion of such a non-trivial function reveals the presence of *only* the energy momentum tensor and the absence of *all* higher-spin currents. This shows how far away supergravity is from a holographic description of perturbative CFTs. This shows also the necessity to consider HS gauge theories if we wish to describe holographically perturbative CFTs.

3 Holography of the critical three-dimensional $O(N)$ vector model

A concrete proposal for the holographic correspondence between a CFT and a HS gauge theory was made in [13]. It was there suggested that the critical three-dimensional $O(N)$ vector model is the holographic dual of the simplest HS gauge theory on AdS$_4$, a theory that contains bosonic symmetric traceless even-rank tensors. The elementary fields of the (Euclidean) three-dimensional $O(N)$ vector model are the scalars

$$\Phi^a(x)\,, \quad a = 1, 2, .., N\,, \tag{14}$$

constrained by

$$\Phi^a(x)\Phi^a(x) = \frac{1}{g}\,. \tag{15}$$

The model approaches a free field theory for $g \to 0$. To calculate the partition function in the presence of sources $J^a(x)$ it is convenient to introduce the Lagrange multiplier field $G(x)$ as

$$Z[J^a] = \int (\mathcal{D}\Phi^a)(\mathcal{D}G)e^{-\frac{1}{2}\int \Phi^a(-\partial^2)\Phi^a + \frac{i}{2}\int G(\Phi^a\Phi^a - \frac{1}{g}) + \int J^a\Phi_a}\,, \tag{16}$$

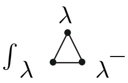
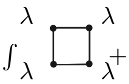
¿From (16) we see that is is natural to consider the effective coupling $\hat{g} = gN$ which for large-N may be adjusted to remain $O(1)$ as $g \to 0$. Integrating out the $\Phi^a s$ and setting $G(x) = G_0 + \lambda(x)/\sqrt{N}$ we obtain the (renormalizable) $1/N$ expansion as

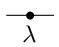
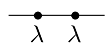
$$\frac{Z[J^a]}{Z_0} = \int (\mathcal{D}\lambda)e^{-\frac{N}{2}[Tr(\ln(1 - \frac{i}{\sqrt{N}}\frac{\lambda}{-\partial^2})) + \frac{i}{\sqrt{N}}\frac{\lambda}{\hat{g}}]}e^{\frac{1}{2}\int J^a \frac{1}{-\partial^2}(1 - \frac{i}{\sqrt{N}}\frac{\lambda}{-\partial^2})^{-1}J^a}\,, \tag{17}$$

where Z_0 depends on G_0. The critical theory is obtained for $G_0 = 0$ and the critical coupling is determined by the gap equation

$$\frac{1}{\hat{g}_*} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2}\,. \tag{18}$$

The resulting generating functional takes the form

$$\frac{Z_*[J^a]}{Z_0} = \int(\mathcal{D}\lambda)e^{-\frac{1}{2}\int \lambda K\lambda \,-\, \frac{i}{3!\sqrt{N}}\int}$$  $$-\frac{1}{8N}\int$$  $$+\cdots$$

$$\times\ e^{\frac{1}{2}\int J^a\Big[}$$ $$+\frac{i}{\sqrt{N}}$$  $$-\frac{1}{N}$$  $$+\cdots\Big]J^a\ . \tag{19}$$

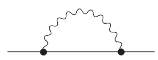
The basic propagator of the $\Phi^a(x)$s is

$$\Delta(x) = \frac{1}{4\pi}\frac{1}{(x^2)^{1/2}} \quad \sim$$ (20)

The operator K and its inverse are then found to be

$$K = \frac{\Delta^2}{2}\,, \quad K^{-1} = -\frac{16}{\pi^2}\frac{1}{x^4} \quad \sim \quad -$$ (21)

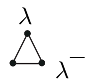
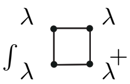
Now we can calculate all n-pt functions of Φ^a. For example, the 2-pt function is given by

$$\langle\Phi^a\Phi^b\rangle = \delta^{ab}\Big[$$ $$-\frac{1}{N}$$  $$+\cdots\Big]$$

$$= \delta^{ab}\frac{1}{(x^2)^{1/2}}[1-\eta_1\ln x^2+...]\,, \qquad \eta_1 = \frac{1}{N}\frac{4}{3\pi^2}\ . \tag{22}$$

We notice that the elementary fields have acquired an anomalous dimension η_1. Ideally, a holographic description of the $O(N)$ vector model should reproduce this result from a bulk calculation, however, such a calculation is still elusive. On the other hand, bulk fields would give the correlation functions of composite boundary operators. The generating functional for one such operator may be obtained if we consider a external source for the fluctuations of the auxiliary field as

$$\frac{Z[J^a]}{Z_0} \to Z[A]$$

$$= \int(\mathcal{D}\lambda)e^{-\frac{1}{2}\int\lambda K\lambda \,-\, \frac{i}{3!\sqrt{N}}\int}$$  $$-\frac{1}{8N}\int$$  $$+\cdots+\int A\lambda\ . \tag{23}$$

This can be viewed as the geenrating functional $e^{\hat{W}[A]}$ for a conformal scalar operator λ with a dimension $\Delta = 2 + O(1/N)$. To be precise, since

$$\int(\mathcal{D}\lambda)e^{-\frac{1}{2}\int\lambda K\lambda + \int A\lambda} = e^{\frac{1}{2}\int A\Pi\lambda}\,, \tag{24}$$

and Π would give a non-positive 2-pt function, we should actually consider

$$W[A] = \hat{W}[iA]\ . \tag{25}$$

The proposal of [13] may then be concretely presented as

$$e^{W[A]} \equiv \int_{AdS_4}(\mathcal{D}\Phi)e^{-I_{HS}(\Phi)}\,, \tag{26}$$

where

$$I_{HS}(\Phi) = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{g}\Big(\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + ...\Big)\,, \tag{27}$$

with $m^2 = -2$ that corresponds to conformally coupled scalar on AdS_4. We see that the problem in hand naturally asks for a "bottom-up" approach, namely, use the full knowledge of the boundary

effective action in order to calculate the bulk path integral. Indeed, in principle we should be able to have control of the fully quantized bulk theory, since bulk quantum corrections would correspond to the $1/N$ corrections of a renormalizable boundary theory. For the time being, however, one can be content if knowledge of the boundary generating functional for composite operators can help her calculate the elusive classical bulk action for a HS gauge theory.

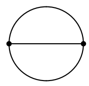
Let me briefly describe now, how this "lifting program" works [14]. A possible form of the bulk HS action is

$$\begin{aligned} I_{HS}(\Phi) &= \frac{1}{2\kappa_4^2}\int d^4x\sqrt{g}\Big(\frac{1}{2}(\partial\Phi)^2-\Phi^2+\frac{g_3}{3!}\Phi^3+\frac{g_4}{4!}\Phi^4 \\ &\quad +\sum_{s=2}^{\infty}\mathcal{G}_s h^{\{\mu_1\dots\mu_s\}}\Phi\,\partial_{\{\mu_1}\dots\partial_{\mu_s\}}\Phi+\cdots\Big)\,, \end{aligned} \tag{28}$$

where $\mathcal{G}_s$ denote the couplings of the conformally coupled scalar to the higher-spin gauge fields $h^{\{\mu_1,\dots,\mu_s\}}$ that can be taken to be totally symmetric traceless and conserved tensors. ¿From (28), by the standard holographic procedure involving the evaluation of the on-shell bulk action with specified boundary conditions, we obtain schematically:

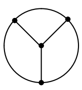
– 2-pt function –

$$\langle\lambda\lambda\rangle \quad \sim \tag{29}$$



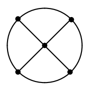
– 3-pt function –

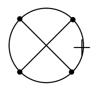
$$\langle\lambda\lambda\lambda\rangle \quad \sim \quad g_3 \tag{30}$$



The 4-pt function depends on g_4, g_3^2 and on *all* the higher-spin couplings $\mathcal{G}_s$. It is technically not impossible to write down the bulk tree graphs that involve the exchange of higher-spin currents [15]. Moreover, bulk gauge invariance should, in principle, be sufficient to determine *all* the $\mathcal{G}_s$ in terms of only one of them i.e. in terms of the coupling $\mathcal{G}_2$ of the energy momentum tensor. Schematically we have

$$\langle\lambda\lambda\lambda\lambda\rangle \quad \sim \quad g_4 \;\; [\text{diagram}] \;\; + \;\; g_3^2\Big[\;[\text{diagram}] + \text{crossed}\Big] + \sum_s \mathcal{G}_s\Big[\cdots\Big] \tag{31}$$





The above must be compared with the corresponding result obtained from the $W[A]$. This would allow us to fix the scalings of various coefficients as (we set the AdS radius to 1)

$$\frac{1}{2\kappa_4^2}\sim N\,,\quad g_3\,,g_4\sim O(1)\,,\quad \mathcal{G}_s^2\sim O(\frac{1}{N})\;\text{ iff }\;\langle h_s\,h_s\rangle\sim O(1)\,. \tag{32}$$

The first result obtained this way was [14]

$$g_3=0\,. \tag{33}$$

This has been confirmed by a direct calculation using the Vasiliev equations on AdS_4 [16]. Further results have been reported in [15], however much more needs to be done to get a satisfactory understanding of the bulk HS action. Another open issue is to reproduce, from bulk loops, the known boundary anomalous dimension of the operator λ. Finally, it remains to be understood [2] how the Higgs mechanism for the bulk HS fields gives rise to the known $1/N$ corrections of the anomalous dimensions of the corresponding boundary higher-spin operators.

[2] See [17] for a recent work on that issue.

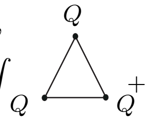
4 The role of boundary double-trace transformations and the first trace of duality

In our study of the critical $O(N)$ vector model we have started with an elementary field Φ^a with dimension $\Delta = 1/2 + O(1/N)$ and obtained a composite operator λ with $\Delta_\lambda = 2 + O(1/N)$. It follows that we are dealing with an interacting CFT even for $N \to \infty$, since the free CFT would have had a composite operator like $\frac{1}{\sqrt{N}}\Phi^a\Phi^a$ with $\Delta_{\Phi^2} = 1$. It is then natural to ask where is the free theory? To find it we consider the Legendre transform of $W[A]$ as

$$W[A] + \int AQ = \Gamma[Q]\,, \tag{34}$$

$$\Gamma[Q] = \Gamma_0[Q] + \frac{1}{N}\Gamma_1[Q] + \ldots\,, \tag{35}$$

$$\Gamma_0[Q] = \frac{1}{2}\int Q\,K\,Q + \frac{1}{3!\sqrt{N}}\int$$



$$+\cdots\,. \tag{36}$$

The generating functional for the correlation functions of the free field $\frac{1}{\sqrt{N}}\Phi^a\Phi^a$ with dimension $\Delta = 1$ is $\Gamma_0[Q]$. Therefore, we face here the holographic description of a free field theory!

The theory described by $\Gamma[Q]$ has imaginary couplings and anomalous dimensions below the unitarity bounds [18]. Nevertheless, *both* the theories described by $W[A]$ and $\Gamma[Q]$ should be the holographic duals of a *unique* HS bulk theory. Moreover, it turns out that these two theories are related to each other by an underlying dynamics that appears to be generic in non-trivial models of three-dimensional CFT. We propose that this particular underlying dynamics is a kind of duality. This is motivated by the fact that for $N \to \infty$ the spectrum of free field theory and the spectrum of theory $W[A]$ are *almost* the same. Indeed, the $W[A]$ theory contains conserved currents for $N \to \infty$. The *only* difference between the two theories at leading-N is the interchange of the two scalar operators corresponding to the two following UIRs of SO(3,2)

$$D(1,0) \longleftrightarrow D(2,0)\,. \tag{37}$$

These UIRs are *equivalent* as they are related by Weyl reflections and have the same Casimirs.

The *duality* (37) is induced by a particular type of dynamics usually referred to as *double-trace deformations*.[3] To show the essence of our proposal, consider an operator $Q(x)$ with a dimension $\Delta = 1$ i.e. an operator in free field theory. Then $Q^2(x)$ is a relevant deformation of a theory and we can consider the deformed 2-pt function as

$$\begin{aligned}&\langle Q(x_1)Q(x_2)\,e^{\frac{f}{2}\int Q^2(x)}\rangle = \langle Q(x_1)Q(x_2)\rangle_f\\ &= \langle Q(x_1)Q(x_2)\rangle_0 + \frac{f}{2}\int d^3x \langle Q(x_1)Q(x_2)Q^2(x)\rangle_0 + \ldots\,.\end{aligned} \tag{38}$$

We now make a large-N factorization assumption such that, for example,

$$\frac{1}{2}\langle Q(x_1)Q(x_2)Q^2(x)\rangle_0 \simeq \langle Q(x_1)Q(x)\rangle_0\langle Q(x_2)Q(x)\rangle_0 + O\left(\frac{1}{N}\right)\,, \tag{39}$$

and similarly for all correlators that appear in (38). We then obtain

$$\begin{aligned}\langle Q(x_1)Q(x_2)\rangle &= \langle Q(x_1)Q(x_2)\rangle_0\\ &\quad + f\int d^3x \langle Q(x_1)Q(x)\rangle_0\langle Q(x_2)Q(x)\rangle + O\left(\frac{1}{N}\right)\,.\end{aligned} \tag{40}$$

[3]We keep this terminology despite the fact that there are no traces taken here.

In momentum space this looks like

$$Q_f(p) = \frac{Q_0(p)}{1 - fQ_0(p)}\,, \quad Q_0(p) \simeq \frac{1}{p}\,. \tag{41}$$

In the infrared, i.e. for small momenta $|p| \ll f$, we find

$$f^2 Q_f(p) = -\frac{f}{1 - \dfrac{1}{fQ_0(p)}} \simeq -f - Q_0^{-1}(p) + \dots\,. \tag{42}$$

If we drop the non-conformal constant f term on the r.h.s. of (42) we obtain the 2-pt function of an operator with dimension $\Delta_f = 2$. We see that the UV dimension $\Delta_0 = 1$ has changed to the IR dimension $\Delta_f = 2$ and that this change is induced by the double-trace deformation.

The above dynamics must be seen in AdS_4. The on-shell bulk action of a conformally coupled scalar, using the standard Poincaré coordinates, is

$$I_\varepsilon = -\frac{1}{2}\frac{1}{\varepsilon^2}\int d^3x\, \Phi(\bar{x};\varepsilon)\partial_r\Phi(\bar{x};r)\Big|_{r=\varepsilon\ll 1}\,. \tag{43}$$

To evaluate it we need to solve the Dirichlet problem

$$(\nabla^2 - 2)\Phi(\bar{x};r) = 0\,, \quad \Phi(\bar{x};r=\infty) = 0\,, \quad \Phi(\bar{x};\varepsilon) = \Phi(\bar{x};\varepsilon)\,. \tag{44}$$

$$\Phi(\bar{x};r) = \int \frac{d^3p}{(2\pi)^3} e^{i\bar{x}\bar{p}}\Phi(\bar{p};\varepsilon)\,\frac{r}{\varepsilon}\,e^{-|p|(r-\varepsilon)}\,. \tag{45}$$

One way to proceed is via the Dirichlet-to-Neumann map [19] that relates the boundary value of a field in a certain manifold $\mathcal{M}$ to its normal derivative at the boundary, e.g.

$$\Phi(x)\Big|_{x\in\partial\mathcal{M}} = f(\bar{x})\,; \quad \hat{\Lambda}f = n^\mu\partial_\mu\Phi(x)\Big|_{x\in\partial\mathcal{M}}\,, \tag{46}$$

wher n^μ is the normal to the boundary vector. Knowledge of the map $\hat{\Lambda}$ allows the reconstruction of the bulk metric. For the conformally coupled scalar we have the remarkably simple expression

$$\partial_r\Phi(\bar{p};r)\Big|_{r=\varepsilon} = (\frac{1}{\varepsilon} - |p|)\Phi(\bar{p};\varepsilon)\,. \tag{47}$$

The terms in parenthesis on the r.h.s. may be viewed as a *generalized Dirichlet-to-Neumann map* since we have taken the boundary to be at $r = \epsilon$. This map may be identified with the r.h.s. of the expansion (42) is we set $f = 1/\epsilon$, such that

$$\frac{1}{\epsilon^2}\left[\hat{\Lambda}_\epsilon(p)\right]^{-1} \sim f^2 Q_f(p) = f^2\frac{Q_0(p)}{1 - fQ_0(p)}\,. \tag{48}$$

We then see that the inversion of the generalized Dirichlet-to-Neumann map for a conformally coupled scalar corresponds to the resummation induced by a double-trace deformation on the free boundary 2-pt function. Notice that the limit $\epsilon \to 0$ drives the boundary theory in the IR.

Let us finally summarize in a pictorial way some of the salient features of the two types of holography discussed above. In Fig.1 we sketch the standard holographic picture for the correspondence of IIB string-SUGRA/$\mathcal{N} = 4$ SYM. In Fig.2 we sketch what we have learned so far regarding the holography of the HS gauge theory on AdS_4.

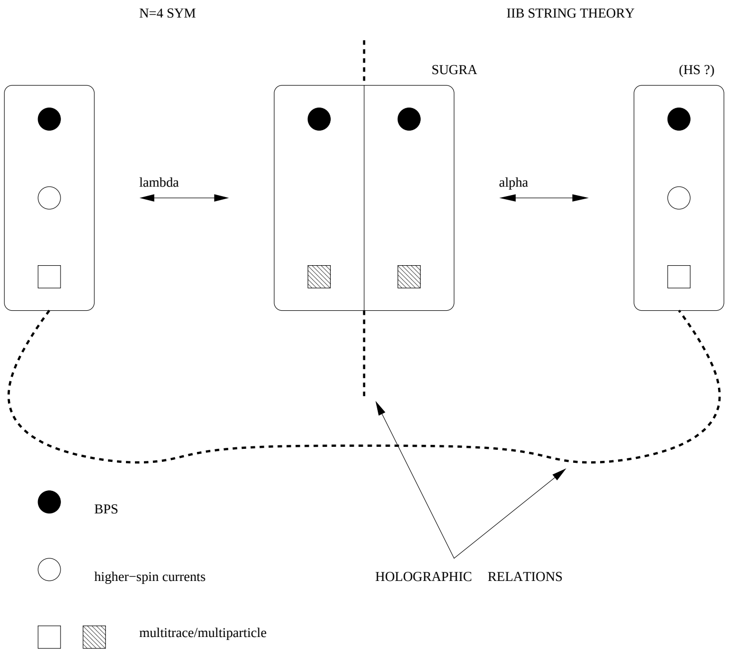


Figure 5.1: Type 1 holographic correspondence: $\mathcal{N} = 4$ SYM/IIB string theory.

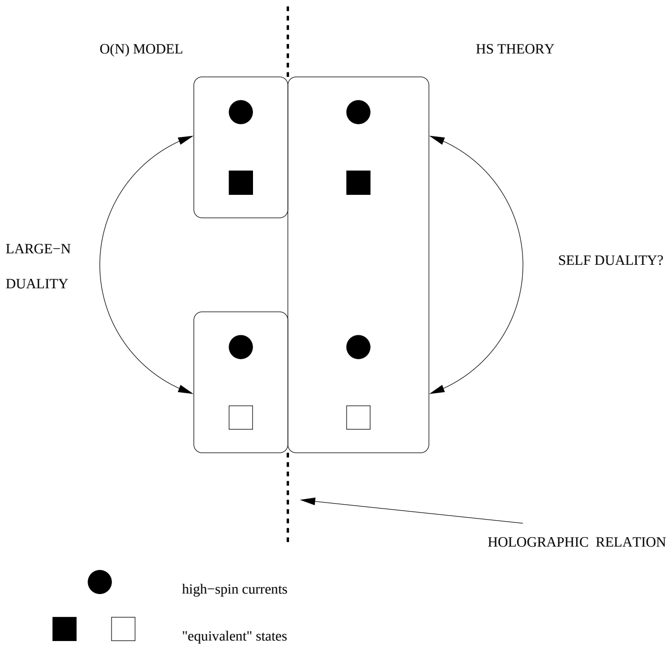


Figure 5.2: Type 2 holographic correspondence: O(N) vector model/HS on AdS_4.

5 Double-trace deformations and the duality of linearized HS gauge theories

In the previous section we emphasized the special role played by the boundary double-trace deformation

$$\frac{f}{2}\int d^3x\, Q(x)Q(x)\,, \tag{49}$$

for theories with bulk conformally invariant scalars. This raises the possibility that boundary deformations of the form

$$\frac{f_s}{2}\int d^3x\, h^{(s)}h_{(s)}\,, \tag{50}$$

where $h^{(s)}$ denote symmetric traceless and conserved currents, may play a crucial role in the holography of HS gauge theories. Such deformations are of course *irrelevant* for all $s \geq 1$, nevertheless in many cases they lead to well-defined UV fixed points. Important examples are the three-dimensional Gross-Neveu and Thirring models in their large-N limits [20]. We should emphasize that the large-N limit is absolutely crucial for the well-defined and non-trivial nature of the above results.

For example, the dynamics underlying the Gross-Neveu model is, in a sense, just the opposite of the dynamics described in the previous section and is again connected with the equivalence or "duality" between the irreps $D(1,0)$ and $D(2,0)$ [21]. However, we face a problem with the idea that the deformations (50) for $s \geq 1$ induce the exchange between equivalent irreps of $SO(3,2)$. Starting with the irreps $D(s+1,s)$, $s \geq 1$, their equivalent irreps are $D(2-s,s)$. These, not only they fall below the unitarity bound $\Delta \geq s+1$, but for $s > 2$ they appear to correspond to operators with negative scaling dimensions. Clearly, such irreps cannot represent physical fields in a QFT. This is to be contrasted with the case of the irreps $D(1,0)$ and $D(2,0)$ which are both abve the unitarity bound.

The remedy of the above problem is suggested by old studies of three- dimensional gauge theories. In particular, it is well-known that to a three-dimensional gauge potential $A_i(p)$ corresponds (we work in momentum space for simplicity), a conserved current $J_i(x) \propto i\epsilon_{ikj}p_jA_k(p)$, while to the three-dimensional spin-2 gauge potential $g_{ij}(p)$ corresponds a conserved current (related to the Cotton tensor [22]), $T_{ij}(p) \propto \Pi^{(1.5)}_{ijkl}g_{kl}(p)$ (see below for the definitions). A similar construction associates to each gauge field belonging to the irrep $D(2-s,s)$ a physical current in the irrep $D(s+1,s)$. Therefore, we need actually two steps to understand the effect of the double-trace deformations (50); firstly the deformation will produce what it looks like a correlator for an operator transforming under $D(2-s,s)$ that may be viewed as a gauge field, secondly we transform this correlator to one of a conserved current $D(s+1,s)$. Moreover, these conserved currents have opposite parity from the gauge fields, so we expect that parity plays a role in our discussion.

Let us be more concrete and see what all the above mean in practice. Consider a boundary CFT with a conserved current J_i having momentum space 2-pt function

$$\langle \mathcal{J}_i\mathcal{J}_i\rangle_0 \equiv (\mathcal{J}_0)_{ij} = \tau_1\frac{1}{|p|}\Pi_{ij} + \tau_2\varepsilon_{ijk}p_k\,, \quad \Pi_{ij} \equiv p_ip_j - \delta_{ij}p^2\,. \tag{51}$$

Consider the irrelevant double-trace deformation

$$\frac{f_1}{2}\int \mathcal{J}_i\mathcal{J}_i\,, \tag{52}$$

and calculate

$$(\mathcal{J}_{f_1})_{ij} \equiv \langle \mathcal{J}_i\mathcal{J}_i\, e^{\frac{f_1}{2}\int \mathcal{J}\mathcal{J}}\rangle = (\mathcal{J}_0)_{ij} + \frac{f_1}{2}\int \langle \mathcal{J}_i\mathcal{J}_j\Big|\mathcal{J}_k\mathcal{J}_k\rangle + \ldots \tag{53}$$

Now assume: i) large-N expansion $\mathcal{J}_i\mathcal{J}_j \sim (\mathcal{J})_{ij} + O(1/N)$ and ii) existence of a UV fixed-point. The leading-N resummation yields

$$f_1^2(\mathcal{J}_{f_1})_{ij} = \hat{\tau}_1\frac{1}{|p|}\Pi_{ij} + \hat{\tau}_2\varepsilon_{ijk}p_k \tag{54}$$

$$\hat{\tau}_1 \simeq \frac{f_1}{|p|} + \frac{1}{|p|^2}\frac{\tau_1}{\tau_1^2+\tau_2^2} + ... \tag{55}$$

$$\hat{\tau}_2 \simeq -\frac{1}{|p|^2}\frac{\tau_2}{\tau_1^2+\tau_2^2} + ... \tag{56}$$

Dropping the non-conformally invariant term $f_1/|p|$ we get

$$f_1^2(\mathcal{J}_{f_1})_{ij} = \frac{\tau_1}{\tau_1^2+\tau_2^2}\frac{1}{|p|^3}\Pi_{ij} - \frac{\tau_2}{\tau_1^2+\tau_2^2}\varepsilon_{ijk}p_k \; . \tag{57}$$

This is the 2-pt function of a conformal operator $\hat{A}_i(\bar{p})$ transforming in the irrep $D(1,1)$. It lies below the unitary bound $\Delta \geq s+1$ of $SO(3,2)$, therefore it must be a gauge field. Define then the current $\hat{\mathcal{J}}_i = i\varepsilon_{ijk}p_j\hat{A}_k$ that has 2-pt function

$$\langle \hat{\mathcal{J}}_i\hat{\mathcal{J}}_j\rangle = \frac{\tau_1}{\tau_1^2+\tau_2^2}\frac{1}{|p|^3}\Pi_{ij} - \frac{\tau_2}{\tau_1^2+\tau_2^2}\varepsilon_{ijk}p_k \; . \tag{58}$$

It follows that there exists a "dual" theory with current $\hat{\mathcal{J}}_i$ that has 2-pt function obtained from (51) by

$$\tau \to -\frac{1}{\tau}\, , \quad \tau = \tau_2 + i\tau_1 \, . \tag{59}$$

Similarly, we may consider a boundary CFT having an energy momentum tensor with 2-pt function

$$\langle T_{ij}T_{kl}\rangle = \kappa_1\frac{1}{|p|}\Pi^{(2)}_{ij,kl} - \kappa_2\Pi^{(1.5)}_{ij,kl} \, , \tag{60}$$

where

$$\begin{aligned}\Pi^{(2)}_{ij,kl} &= \frac{1}{2}\Big[\Pi_{ik}\Pi_{jl} + \Pi_{il}\Pi_{jk} - \Pi_{ij}\Pi_{kl}\Big] \, , \\ \Pi^{(1.5)}_{ij,kl} &= \frac{1}{4}\Big[\varepsilon_{ikp}\Pi_{jl} + \varepsilon_{jkp}\Pi_{il} + \varepsilon_{ilp}\Pi_{jk} + \varepsilon_{jlp}\Pi_{ik}\Big] \, .\end{aligned} \tag{61}$$

The boundary irrelevant "double-trace" deformation

$$\frac{f_2}{2}\int T_{ij}T_{ij} \, , \tag{62}$$

leads (under the same large-N, existence of UV fixed point assumptions as above), to a theory with an energy momentum tensor that has 2-pt function obtained from (60) by [23]

$$\kappa \to -\frac{1}{\kappa}\, , \;\; \kappa = \kappa_2 + i\kappa_1 \, . \tag{63}$$

It is not difficult to imagine that the picture above generalizes to all higher-spin currents in a three-dimensional CFT.

Now let us discuss what all the above boundary properties mean for the bulk HS gauge theory. The first thing to notice is of course that the form of the transformations (59) and (63) is reminiscent of S-duality transformations. Then, we must ask what could be the bulk action that yields (51) and (60). For example, (51) may be the on-shell boundary value of the $U(1)$ action with a θ-term on AdS$_4$

$$I = \frac{1}{8\pi}\int d^4x\sqrt{g}\Big(\frac{4\pi}{e^2}F_{\mu\nu}F^{\mu\nu} + i\frac{\theta}{2\pi}\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\Big) \, . \tag{64}$$

We will show now that the bulk dual of the double-trace boundary deformation discussed previously is a canonical duality transformation i.e. a canonical transformation that interchanges the bulk canonical

variables. We use the ADM form of the Euclidean AdS$_4$ metric (with radius set to 1)

$$ds^2 = d\rho^2 + \gamma_{ij}dx^i dx^j\ , \quad \gamma_{ij} = e^{2\rho}\eta_{ij}\ , \ \ \gamma = \mathrm{det}\gamma_{ij}\ , \ \ i,j = 1,2,3\,. \tag{65}$$

to write the bulk action (64) in terms of the canonical variables as

$$I = \int d\rho\, d^3x\, \sqrt{\gamma}\left[\Pi^i \dot{A}_i - \mathcal{H}(\Pi^i, A_i)\right]\ , \tag{66}$$

$$\mathcal{H}(\Pi^1, A_i) = \frac{1}{e^2}\gamma^{-1}\gamma_{ij}\left(\mathcal{E}^i\mathcal{E}^j - \mathcal{B}^i\mathcal{B}^j\right)\ , \tag{67}$$

$$\sqrt{\gamma}\Pi^i = \frac{2}{e^2}\mathcal{E}^i + i\frac{\theta}{4\pi^2}\mathcal{B}^i\ , \ \ \mathcal{E}^i = \sqrt{\gamma}E^i\ , \ \mathcal{B}^i = \sqrt{g}B^i\ , \tag{68}$$

with $E^i = F^{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ the usual electric and magnetic fields. The essence of the Hamilton-Jacobi approach to AdS holography is that for a given ρ_0 the variation of the bulk action with respect to the canonical variable A_i gives, on shell, the canonical momentum Π^i at ρ_0. For $\rho_0 \to \infty$ the latter is interpreted as the *regularized* 1-pt function in the presence of sources, the reason being that requiring the regularity of the classcial solutions inside AdS gives a relation between Π^i and A_i. Finally, to reach the boundary one invokes a further technical step, (sometimes called holographic renormalization), such as to obtain finite 1-pt functions from which all correlation functions of the boundary CFT can be found. Schematically we have

$$\frac{1}{\sqrt{\gamma}}\,\frac{\delta I}{\delta A_i(\rho_0,x_i)}\bigg|_{on-shell} = \Pi^i(\rho_0,x_i) \sim_{\rho_0\to\infty} \langle\mathcal{J}^i(x_i)\rangle_{A_i} \tag{69}$$

In fact, one can show that for a $U(1)$ field on AdS$_4$ there is no need for renormalization as the solutions of the bulk e.o.m. give finite contributions at the boundary.

Next we consider canonical transformations in the bulk to move from the set variables (A_i, Π^i) to the new set $(\tilde{A}_i, \tilde{\Pi}^i)$. In particular, we may consider a generating functional of the 1st kind (see e.g. [25]) of the form

$$\mathcal{F}[A_i, \tilde{A}_i] = \frac{1}{2}\int_{\rho=\,fixed} d^3x\ \sqrt{\gamma}A_i(\rho,x_i)\epsilon^{ijk}\tilde{F}_{jk}(\rho,x_i)\,. \tag{70}$$

This induces the transformations

$$\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{F}}{\delta A_i} \equiv \Pi^i = \tilde{B}^i\ , \tag{71}$$

$$\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{F}}{\delta \tilde{A}_i} \equiv -\tilde{\Pi}^i = B^i\,. \tag{72}$$

For $\theta = 0$ these are the standard duality transformations $E^i \to B^i$, $B^i \to -E^i$. We have at our disposal now two bulk actions, one written in terms of (A_i, Π^i) and the other in terms of $(\tilde{A}_i, \tilde{\Pi}^i)$, which according to (69) give at $\rho = \infty$,

$$\langle\mathcal{J}_i\rangle_{A_i} = \tilde{B}_i = i\epsilon_{ijk}p_j\tilde{A}_k\ , \tag{73}$$

$$\langle\tilde{\mathcal{J}}_i\rangle_{\tilde{A}_i} = -B_i = -i\epsilon_{ijk}p_jA_k\,. \tag{74}$$

¿From the above 1-pt functions we can calculate the corresponding 2-pt functions by functionally differentiating with respect to A_i and $\tilde{A}_i$. We now take the following ansatz for the the matrix $\delta\tilde{A}_i/\delta A_j$

$$\frac{\delta\tilde{A}_i}{\delta A_j} = C_1\frac{1}{p^2}\Pi_{ij} + C_2\epsilon_{ijk}\frac{p_k}{|p|} + (\xi-1)\frac{p_ip_j}{p^2}\ , \tag{75}$$

where ξ plays as usual the role of gauge fixing, neccesary for its inversion. Then from (73) and (74) we find, independently of ξ

$$\langle\mathcal{J}_i\mathcal{J}_k\rangle\langle\tilde{\mathcal{J}}_k\tilde{\mathcal{J}}_j\rangle = -\Pi_{ij}\ , \tag{76}$$

with Π_{ij} defined in (51). It is then easy to verify that if $\langle \mathcal{J}_i \mathcal{J}_k \rangle$ is given by the r.h.s. of (51), $\langle \tilde{\mathcal{J}}_k \tilde{\mathcal{J}}_j \rangle$ is given by the r.h.s. of (58). In other words, we have shown that the bulk canonical transformation generated by (70) induces the S-transformation (59) on the boundary 2-pt functions. We expect that our result generalizes to linearized bulk gravity in the Hamiltonian formalism[4] and linearized higher-spin theories.

An intriguing property of the above transformations generated by the boundary double-trace deformations is that combined with the trivial transformation defined as

$$\tau \to \tau + 1\,, \tag{77}$$

form the SL(2,Z) group [24]. The transformation (77) is the boundary image of the bulk shift of the θ-angle

$$\theta \to \theta + 2\pi\,. \tag{78}$$

We expect that an analogous effect takes place in linearized higher-spin gauge theories on AdS_4 when the appropriate θ-terms are introduced in the bulk [23, 27]. The above suggest a special role for the SL(2,Z) group in the study of HS gauge theories, even at the quantum level.

6 Discussion

It has been suggested [28] that HS gauge theories emerge at the tensionless limit of string theory, in a way similar to the emergence of supergravity at the limit of infinite string tension. It would be extremely interesting to quantify the above statement. A step in this direction is the study of the holography of higher-spin theories on AdS spaces. In this direction, both the study of specific models, as well as investigations of generic holographic properties of HS theories are important. The three-dimensional $O(N)$ vector model provides a concrete example of a theory with a holographic HS dual. On the other hand, bulk dualities of linearized higher-spin theories may have far reaching consequences for their CFT duals. For example, if a theory posesses a HS dual with a self-duality property, its boundary double-trace deformations despite being irrelevant might lead to a well-defined UV completion of the theory. Also, it is conceivable that the self-duality property of linearized HS theories is a remnant of a string theory duality in the tensionless limit. Finally, it is well-known that 2-pt functions of three-dimensional spin-1 conserved currents can describe observable properties of Quantum Hall systems [29]. It is then interesting to ask whether boundary correlation functions of higher-spin currents may describe observables properties of physical systems. In particular, it is intriguing to suggest [30] that linear gravity in AdS_4 may correspond to special kinds of three-dimensional fluids in which SL(2,Z) or a subgroup of it play a role.

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An Introduction to Free Higher-Spin Fields

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Abstract. In this article we begin by reviewing the (Fang-)Fronsdal construction and the non-local geometric equations with unconstrained gauge fields and parameters built by Francia and the senior author from the higher-spin curvatures of de Wit and Freedman. We then turn to the triplet structure of totally symmetric tensors that emerges from free String Field Theory in the $\alpha' \to 0$ limit and its generalization to (A)dS backgrounds, and conclude with a discussion of the simple local compensator form of the field equations that displays the unconstrained gauge symmetry of the non-local equations.

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1 Introduction

This article reviews some of the developments that led to the free higher-spin equations introduced by Fang and Fronsdal [1, 2] and the recent constructions of free non-local geometric equations and local compensator forms of [3, 4, 5]. It is based on the lectures delivered by A. Sagnotti at the First Solvay Workshop, held in Brussels on May 2004, carefully edited by the other authors for the online Proceedings.

The theory of particles of arbitrary spin was initiated by Fierz and Pauli in 1939 [6], that followed a field theoretical approach, requiring Lorentz invariance and positivity of the energy. After the works of Wigner [7] on representations of the Poincaré group, and of Bargmann and Wigner [8] on relativistic field equations, it became clear that the positivity of energy could be replaced by the requirement that the one-particle states carry a unitary representation of the Poincaré group. For massive fields of integer and half-integer spin represented by totally symmetric tensors $\Phi_{\mu_1\ldots\mu_s}$ and $\Psi_{\mu_1\ldots\mu_s}$, the former requirements are encoded in the Fierz-Pauli conditions

$$(\Box - M^2)\Phi_{\mu_1\ldots\mu_s} = 0 , \qquad (i\partial\!\!\!/ - M)\Psi_{\mu_1\ldots\mu_s} = 0 , \tag{1}$$

$$\partial^{\mu_1}\Phi_{\mu_1\ldots\mu_s} = 0 , \qquad \partial^{\mu_1}\Psi_{\mu_1\ldots\mu_s} = 0 . \tag{2}$$

The massive field representations are also irreducible when a (γ-)trace condition

$$\eta^{\mu_1\mu_2}\Phi_{\mu_1\mu_2\ldots\mu_s} = 0 , \qquad \gamma^{\mu_1}\Psi_{\mu_1\ldots\mu_s} = 0 . \tag{3}$$

is imposed on the fields.

A Lagrangian formulation for these massive spin s-fields was first obtained in 1974 by Singh and Hagen [9], introducing a sequence of auxiliary traceless fields of ranks $s-2$, $s-3$, $\ldots 0$ or 1, all forced to vanish when the field equations are satisfied.

Studying the corresponding massless limit, in 1978 Fronsdal obtained [1] four-dimensional covariant Lagrangians for massless fields of any integer spin. In this limit, all the auxiliary fields decouple and may be ignored, with the only exception of the field of rank $s-2$, while the two remaining traceless tensors of rank s and $s-2$ can be combined into a single tensor $\varphi_{\mu_1\cdots\mu_s}$ subject to the unusual "double trace" condition

$$\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\varphi_{\mu_1\cdots\mu_s} = 0 . \tag{4}$$

Fang and Fronsdal [2] then extended the result to half-integer spins subject to the peculiar "triple γ trace" condition

$$\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\psi_{\mu_1\cdots\mu_s} = 0 . \tag{5}$$

It should be noted that the description of massless fields in four dimensions is particularly simple, since the massless irreducible representations of the Lorentz group SO(3,1)$^{\uparrow}$ are exhausted by totally symmetric tensors. On the other hand, it is quite familiar from supergravity [10] that in dimensions $d > 4$ the totally symmetric tensor representations do not exhaust all possibilities, and must be supplemented by mixed ones. For the sake of simplicity, in this paper we shall confine our attention to the totally symmetric case, focussing on the results of [3, 4, 5]. The extension to the mixed-symmetry case was originally obtained in [11], and will be reviewed by C. Hull in his contribution to these Proceedings [12].

2 From Fierz-Pauli to Fronsdal

This section is devoted to some comments on the conceptual steps that led to the Fronsdal [1] and Fang-Fronsdal [2] formulations of the free high-spin equations. As a first step, we describe the salient features of the Singh-Hagen construction of the massive free field Lagrangians [9]. For simplicity, we shall actually refer to spin 1 and 2 fields that are to satisfy the Fierz-Pauli conditions (1)-(2), whose equations are of course known since Maxwell and Einstein. The spin-1 case is very simple, but the spin-2 case already presents the key subtlety. The massless limit will then illustrate the simplest instances of Fronsdal gauge symmetries. The Kaluza-Klein mechanism will be also briefly discussed, since it exhibits rather neatly the rationale behind the Fang-Fronsdal auxiliary fields for the general case. We then turn

to the novel features encountered with spin 3 fields, before describing the general Fronsdal equations for massless spin-s bosonic fields. The Section ends with the extension to half-integer spins.

2.1 Fierz-Pauli conditions

Let us first introduce a convenient compact notation. Given a totally symmetric tensor φ, we shall denote by $\partial\varphi$, $\partial\cdot\varphi$ and φ' (or, more generally, $\varphi^{[p]}$) its gradient, its divergence and its trace (or its p-th trace), with the understanding that in all cases the implicit indices are totally symmetrized.

Singh and Hagen [9] constructed explicitly Lagrangians for spin-s fields that give the correct Fierz-Pauli conditions. For spin 1 fields, their prescription reduces to the Lagrangian

$$\mathcal{L}_{spin1} = -\frac{1}{2}(\partial_\mu\Phi_\nu)^2 - \frac{1}{2}(\partial\cdot\Phi)^2 - \frac{M^2}{2}(\Phi_\mu)^2 \ , \tag{6}$$

that gives the Proca equation

$$\Box\Phi_\mu - \partial_\mu(\partial\cdot\Phi) - M^2\Phi_\mu = 0 \ . \tag{7}$$

Taking the divergence of this field equation, one obtains immediately $\partial^\mu\Phi_\mu = 0$, the Fierz-Pauli transversality condition (2), and hence the Klein-Gordon equation for Φ_μ.

In order to generalize this result to spin-2 fields, one can begin from

$$\mathcal{L}_{spin2} = -\frac{1}{2}(\partial_\mu\Phi_{\nu\rho})^2 + \frac{\alpha}{2}(\partial\cdot\Phi_\nu)^2 - \frac{M^2}{2}(\Phi_{\mu\nu})^2 \ , \tag{8}$$

where the field $\Phi_{\mu\nu}$ is traceless. The corresponding equation of motion reads

$$\Box\Phi_{\mu\nu} - \frac{\alpha}{2}\left(\partial_\mu\partial\cdot\Phi_\nu + \partial_\nu\partial\cdot\Phi_\mu - \frac{2}{D}\,\eta_{\mu\nu}\,\partial\cdot\partial\cdot\Phi\right) - M^2\Phi_{\mu\nu} = 0 \ , \tag{9}$$

whose divergence implies

$$\left(1-\frac{\alpha}{2}\right)\Box\partial\cdot\Phi_\nu + \alpha\left(-\frac{1}{2}+\frac{1}{D}\right)\partial_\nu\partial\cdot\partial\cdot\Phi - M^2\partial\cdot\Phi_\nu = 0 \ . \tag{10}$$

Notice that, in deriving these equations, we have made an essential use of the condition that Φ be traceless.

In sharp contrast with the spin 1 case, however, notice that now the transversality condition is not recovered. Choosing $\alpha = 2$ would eliminate some terms, but one would still need the additional constraint $\partial\cdot\partial\cdot\Phi = 0$. Since this is not a consequence of the field equations, the naive system described by Φ and equipped with the Lagrangian $\mathcal{L}_{spin2}$ is unable to describe the free spin 2 field.

One can cure the problem introducing an auxiliary scalar field π in such a way that the condition $\partial\cdot\partial\cdot\Phi = 0$ be a consequence of the Lagrangian. Let us see how this is the case, and add to (8) the term

$$\mathcal{L}_{add} = \pi\,\partial\cdot\partial\cdot\Phi + c_1(\partial_\mu\pi)^2 + c_2\pi^2 \ , \tag{11}$$

where $c_{1,2}$ are a pair of constants. Taking twice the divergence of the resulting equation for $\Phi_{\mu\nu}$ gives

$$\left[(2-D)\,\Box - DM^2\right]\partial\cdot\partial\cdot\Phi + (D-1)\,\Box^2\pi = 0 \ , \tag{12}$$

while the equation for the auxiliary scalar field reduces to

$$\partial\cdot\partial\cdot\Phi + 2(c_2 - c_1\Box)\pi \ = \ 0 \ . \tag{13}$$

Eqs. (12) and (13) can be regarded as a linear homogeneous system in the variables $\partial\cdot\partial\cdot\Phi$ and π. If the associated determinant never vanishes, the only solution will be precisely the missing condition $\partial\cdot\partial\cdot\Phi = 0$, together with the condition that the auxiliary field vanish as well, $\pi = 0$, and as a result the transversality condition will be recovered. The coefficients c_1 and c_2 are thus determined by the

condition that the determinant of the system

$$\begin{aligned}\Delta &= -2DM^2c_2 + 2((2-D)c_2 + DM^2c_1)\Box &(14)\\ &- (2(2-D)c_1 - (D-1))\Box\Box &(15)\end{aligned}$$

be algebraic, *i.e.* proportional to the mass M but without any occurrence of the D'Alembert operator $\Box$. Hence, for $D > 2$,

$$c_1 = \frac{(D-1)}{2(D-2)} \ , \qquad c_2 = \frac{M^2D(D-1)}{2(D-2)^2} \ . \tag{16}$$

The end conclusion is that the complete equations imply

$$\pi = 0 \ , \qquad \partial\cdot\partial\cdot\Phi = 0 \ , \tag{17}$$

$$\partial\cdot\Phi_\nu = 0 \ , \qquad \Box\Phi_{\mu\nu} - M^2\Phi_{\mu\nu} = 0 \ , \tag{18}$$

the Fierz-Pauli conditions (1) and (2), so that the inclusion of a single auxiliary scalar field leads to an off-shell formulation of the free massive spin-2 field.

2.2 "Fronsdal" equation for spin 2

We can now take the $M \to 0$ limit, following in spirit the original work of Fronsdal [1]. The total Lagrangian $\mathcal{L}_{spin2} + \mathcal{L}_{add}$ then becomes

$$\mathcal{L} \ = \ -\frac{1}{2}\left(\partial_\mu\Phi_{\nu\rho}\right)^2 + \left(\partial\cdot\Phi_\nu\right)^2 + \pi\partial\cdot\partial\cdot\Phi + \frac{D-1}{2(D-2)}\left(\partial_\mu\pi\right)^2 \ , \tag{19}$$

whose equations of motion are

$$\Box\Phi_{\mu\nu} - \partial_\mu\partial\cdot\Phi_\nu - \partial_\nu\partial\cdot\Phi_\mu + \frac{2}{D}\,\eta_{\mu\nu}\,\partial\cdot\partial\cdot\Phi + \partial_\mu\partial_\nu\pi = 0 \ , \tag{20}$$

$$\frac{D-1}{D-2}\,\Box\pi - \partial\cdot\partial\cdot\Phi = 0 \ . \tag{21}$$

The representation of the massless spin 2-gauge field via a traceless two-tensor $\Phi_{\mu\nu}$ and a scalar π may seem a bit unusual. In fact, they are just an unfamiliar basis of fields, and the linearized Einstein gravity in its standard form is simply recovered once they are combined in the unconstrained two-tensor

$$\varphi_{\mu\nu} = \Phi_{\mu\nu} + \frac{1}{D-2}\,\eta_{\mu\nu}\,\pi \ . \tag{22}$$

In terms of $\varphi_{\mu\nu}$, the field equations and the corresponding gauge transformations then become

$$\mathcal{F}_{\mu\nu} \equiv \Box\varphi_{\mu\nu} - (\partial_\mu\partial\cdot\varphi_\nu + \partial_\nu\partial\cdot\varphi_\mu) + \partial_\mu\partial_\nu\varphi' = 0 \ , \tag{23}$$

$$\delta\varphi_{\mu\nu} = \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu \ , \tag{24}$$

that are precisely the linearized Einstein equations, where the "Fronsdal operator" $\mathcal{F}_{\mu\nu}$ is just the familiar Ricci tensor. The corresponding Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\varphi_{\nu\rho}\right)^2 + \left(\partial\cdot\varphi_\mu\right)^2 + \frac{1}{2}\left(\partial_\mu\varphi'\right)^2 + \varphi'\partial\cdot\partial\cdot\varphi \ , \tag{25}$$

and yields the Einstein equations $\mathcal{F}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{F}' = 0$, that only when combined with their trace imply the previous equation, $\mathcal{F}_{\mu\nu} = 0$.

The massless case is very interesting by itself, since it exhibits a relatively simple instance of gauge symmetry, but also for deducing the corresponding massive field equations via a proper Kaluza-Klein reduction. This construction, first discussed in [13], is actually far simpler than the original one of [9] and gives a rationale to their choice of auxiliary fields.

Let us content ourselves with illustrating the Kaluza-Klein mechanism for spin 1 fields. To this end, let us introduce a field[1] A_M living in $D+1$ dimensions, that decomposes as $A_M = (A_\mu(x,y), \pi(x,y))$, where y denotes the coordinate along the extra dimension. One can expand these functions in Fourier modes in y and a single massive mode corresponding to the D-dimensional mass m, letting for instance $A_M = (A_\mu(x), -i\pi(x)) \exp imy$ where the judicious insertion of the factor $-i$ will ensure that the field π be real. The $D+1$-dimensional equation of motion and gauge transformation

$$\Box A_M - \partial_M \partial \cdot A = 0 \ , \tag{26}$$

$$\delta A_M = \partial_M \Lambda \tag{27}$$

then determine the D-dimensional equations

$$\begin{aligned}
&\left(\Box - m^2\right) A_\mu - \partial_\mu \left(\partial \cdot A + m\pi\right) = 0, \\
&\left(\Box - m^2\right) \pi + m \left(\partial \cdot A + m\pi\right) = 0 \ , \\
&\delta A_\mu = \partial_\mu \Lambda, \qquad \delta\pi = -m\Lambda \ ,
\end{aligned} \tag{28}$$

where the leftover massive gauge symmetry, known as a Stueckelberg symmetry, is inherited from the higher dimensional gauge symmetry. Fixing the gauge so that $\pi = 0$, one can finally recover the Proca equation (7) for A_μ. The spin-2 case is similar, and the proper choice is $\varphi_{MN}(\varphi_{\mu\nu}, -i\varphi_\mu, -\varphi) \exp imy$, so that the resulting gauge transformations read

$$\begin{aligned}
&\delta\varphi_{MN} = \partial_M \Lambda_N + \partial_N \Lambda_M \ , \\
&\delta\varphi_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \ , \\
&\delta\varphi_\mu = \partial_\mu \Lambda - m\Lambda_\mu \ , \\
&\delta\varphi = -2m\Lambda \ .
\end{aligned} \tag{29}$$

In conclusion, everything works as expected when the spin is lower than or equal to two, and the Fierz-Pauli conditions can be easily recovered. However, some novelties do indeed arise when then spin becomes higher than two.

Let us try to generalize the theory to spin-3 fields by insisting on the equations

$$\mathcal{F}_{\mu\nu\rho} \equiv \Box\varphi_{\mu\nu\rho} - (\partial_\mu \partial \cdot \varphi_{\nu\rho} + perm\,) + (\partial_\mu \partial_\nu \varphi'_\rho + perm) = 0 \ , \tag{30}$$

$$\delta\varphi_{\mu\nu\rho} = \partial_\mu \Lambda_{\nu\rho} + \partial_\nu \Lambda_{\rho\mu} + \partial_\rho \Lambda_{\mu\nu} \ , \tag{31}$$

that follow the same pattern, where *perm* denotes cyclic permutations of $\mu\nu\rho$. In this formulation, there are no auxiliary fields and the trace φ' of the gauge field does not vanish.

Let us first remark that, under a gauge transformation, $\mathcal{F}$ transforms according to

$$\delta\mathcal{F}_{\mu_1\mu_2\mu_3} = 3\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\Lambda' \ . \tag{32}$$

Therefore, *$\mathcal{F}$ is gauge invariant if and only if the gauge parameter is traceless,* $\Lambda' = 0$. This rather unnatural condition will recur systematically for all higher spins, and will constitute a drawback of the Fronsdal formulation.

We can also see rather neatly the obstruction to a geometric gauge symmetry of the spin-3 Fronsdal Lagrangian along the way inspired by General Relativity. Indeed, the spin-3 Fronsdal equation differs in a simple but profound way from the two previous cases, since both for spin 1 and for spin 2 *all lower-spin constructs* built out of the gauge fields are present, while for spin 3 only constructs of spin 3 ($\varphi_{\mu\nu\rho}$), spin 2 ($\partial \cdot \varphi_{\mu\nu}$) and spin 1 ($\varphi'_\mu$) are present. Actually, de Wit and Freedman [14] classified long ago the higher-spin analogs of the spin-2 Christoffel connection $\Gamma_{mu;\nu_1\nu_2}$: they are a hierarchy of connections $\Gamma_{\mu_1\ldots\mu_k;\nu_1\nu_2\nu_3}$, with $k = 1, \ldots (s-1)$, that contain k derivatives of the gauge field. They

[1]Capital Latin letters denote here indices in $D+1$ dimensions, while Greek letters denote the conventional ones in D dimensions.

also noticed that the analog of the Riemann tensor, that for spin 3 would be $\Gamma_{\mu_1\mu_2\mu_3;\nu_1\nu_2\nu_3}$, would in general contain s derivatives of the gauge field, and *related the Fronsdal operator $\mathcal{F}$ to the trace of the second connection.*

One way to bypass the problem is to construct a field equation with two derivatives depending on the true Einstein tensor for higher spins, that however, as we have anticipated, contains s derivatives in the general case. This can be achieved, but requires that non-local terms be included both in the field equations and in the Lagrangian. This approach will be further developed in section 3. Another option is to compensate the non-vanishing term in the right-hand side of (32) by introducing a new field, a *compensator*. As we shall see, this possibility is actually suggested by String Theory, and will be explained in section 4.3.

But before describing these new methods, let us first describe the general Fronsdal formulation for arbitrary spin. In this way we shall clearly identify the key role of the *trace condition on the gauge parameter* that we already encountered for spin 3 and of the *double trace condition on the field*, that will first present itself for spin 4.

2.3 Fronsdal equations for arbitrary integer spin

We can now generalize the reasoning of the previous paragraph to arbitrary integer spins. Since we shall use extensively the compact notation of [3], omitting all indices, is it useful to recall the following rules:

$$\begin{aligned}
(\partial^{\,p}\,\varphi)^{\,\prime} &= \Box\,\partial^{\,p-2}\,\varphi \,+\, 2\,\partial^{\,p-1}\,\partial\cdot\varphi \,+\, \partial^{\,p}\,\varphi^{\,\prime}\,, \\
\partial^{\,p}\,\partial^{\,q} &= \binom{p+q}{p}\,\partial^{\,p+q}\,, \\
\partial\cdot(\partial^{\,p}\,\varphi) &= \Box\,\partial^{\,p-1}\,\varphi \,+\, \partial^{\,p}\,\partial\cdot\varphi\,, \\
\partial\cdot\eta^{\,k} &= \partial\,\eta^{\,k-1}, \\
\left(\eta^k\,T_{(s)}\right)^{\prime} &= k\,\left[\mathcal{D} \,+\, 2(s+k-1)\right]\eta^{k-1}\,T_{(s)} \,+\, \eta^k\,T^{\prime}_{(s)}\,.
\end{aligned} \tag{33}$$

In this compact form, the generic Fronsdal equation and its gauge transformations read simply

$$\mathcal{F} \equiv \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi' = 0\,, \tag{34}$$

$$\delta\varphi = \partial\Lambda\,. \tag{35}$$

In order to find the effect of the gauge transformations on the Fronsdal operators $\mathcal{F}$, one must compute the terms

$$\begin{aligned}
\delta(\partial\cdot\varphi) &= \Box\Lambda + \partial\partial\cdot\Lambda\,, \\
\delta\varphi' &= 2\partial\cdot\Lambda + \partial\Lambda'\,,
\end{aligned}$$

and the result is, for arbitrary spin,

$$\delta\mathcal{F} = \Box(\partial\Lambda) - \partial(\Box\Lambda + \partial\partial\cdot\Lambda) + \partial^2(2\partial\cdot\Lambda + \partial\Lambda') = 3\partial^3\Lambda'\,, \tag{36}$$

where we used $-\partial(\partial\partial\cdot\Lambda) = -2\partial^2(\partial\cdot\Lambda)$, and $\partial^2\partial\Lambda' = 3\partial^3\Lambda'$. Therefore, in all cases where Λ' does not vanish identically, *i.e.* for spin $s \geq 3$, the gauge invariance of the equations requires that the gauge parameter be traceless

$$\Lambda' = 0\,. \tag{37}$$

As second step, one can derive the Bianchi identities for all spins computing the terms

$$\begin{aligned}
\partial\cdot\mathcal{F} &= \Box\partial\varphi' - \partial\partial\cdot\partial\cdot\varphi + \partial^2\partial\cdot\varphi', \\
\mathcal{F}' &= 2\Box\varphi' - 2\partial\cdot\partial\cdot\varphi + \partial^2\varphi'' + \partial\partial\cdot\varphi', \\
\partial\mathcal{F}' &= 2\Box\partial\varphi' - 2\partial\,\partial\cdot\partial\cdot\varphi + 3\partial^3\varphi'' + 2\partial^2\partial\cdot\varphi'.
\end{aligned}$$

Therefore, the Fronsdal operator $\mathcal{F}$ satisfies in general the "anomalous" Bianchi identities

$$\partial \cdot \mathcal{F} - \frac{1}{2}\partial \mathcal{F}' = -\frac{3}{2}\partial^3 \varphi'' \ , \tag{38}$$

where the additional term on the right first shows up for spin $s = 4$. In the Fronsdal construction, one is thus led to impose the constraint $\varphi'' = 0$ for spins $s \geq 4$, since the Lagrangians would vary according to

$$\delta\mathcal{L} \ = \ \delta\varphi \left(\mathcal{F} - \frac{1}{2}\eta\mathcal{F}' \right) \ , \tag{39}$$

that does not vanish unless the double trace of φ vanishes identically. Indeed,

$$\partial \cdot \left(\mathcal{F} - \frac{1}{2}\mathcal{F}' \right) = -\frac{3}{2}\partial^3 \varphi'' - \frac{1}{2}\eta \partial \cdot \mathcal{F}' \ , \tag{40}$$

where the last term gives a vanishing contribution to $\delta\mathcal{L}$ if the parameter Λ is traceless. To reiterate, this relation is at the heart of the usual restrictions present in the Fronsdal formulation to traceless gauge parameters and doubly traceless fields, needed to ensure that the variation of the lagrangian

$$\delta\mathcal{L} = \delta\varphi \, \mathcal{G}, \tag{41}$$

where $\mathcal{G} = \mathcal{F} - \frac{1}{2}\eta\mathcal{F}'$, vanishes for $\delta\varphi = \partial\Lambda$.

We can also extend the Kaluza-Klein construction to the spin-s case. Given the double trace condition $\varphi'' = 0$, the reduction $\varphi^{(s)}_{D+1}$ from $D+1$ dimensions to D dimensions gives rise to the tensors $\varphi^{(s)}_{D}$, ..., $\varphi^{(s-3)}_{D}$ of rank s to $s-3$ only. In addition, the trace condition on the gauge parameter implies that only two tensors $\Lambda^{(s-1)}_D$ and $\Lambda^{(s-2)}_D$ are generated in D dimensions. Gauge fixing the Stueckelberg symmetries, one is left with only two traceful fields $\varphi^{(s)}_{D}$ and $\varphi^{(s-3)}_{D}$. But a traceful spin-s tensor contains traceless tensors of ranks s, $s-2$, $s-4$, etc. Hence, the two remaining fields $\varphi^{(s)}_{D}$ and $\varphi^{(s-3)}_{D}$ contain precisely the tensors introduced by Singh and Hagen [9]: a single traceless tensor for all ranks from s down to zero, with the only exception of the rank-$(s-1)$ tensor, that is missing.

2.4 Massless fields of half-integer spin

Let us now turn to the fermionic fields, and for simplicity let us begin with the Rarita-Schwinger equation [15], familiar from supergravity [16]

$$\gamma^{\mu\nu\rho} \, \partial_\nu \psi_\rho = 0 \ , \tag{42}$$

that is invariant under the gauge transformation

$$\delta\psi_\mu = \partial_\mu \epsilon \ , \tag{43}$$

where $\gamma^{\mu\nu\rho}$ denotes the fully antisymmetric product of three γ matrices, normalized to their product when they are all different:

$$\gamma^{\mu\nu\rho} \ = \ \gamma^\mu \gamma^{\nu\rho} - \eta^{\mu\nu}\gamma^\rho + \eta^{\mu\rho}\gamma^\nu \ . \tag{44}$$

Contracting the Rarita-Schwinger equation with γ_μ yields

$$\gamma^{\nu\rho}\partial_\nu \psi_\rho = 0 \ , \tag{45}$$

and therefore the field equation for spin 3/2 can be written in the alternative form

$$\not{\partial}\psi_\mu - \partial_\mu \not{\psi} = 0 \ . \tag{46}$$

Let us try to obtain a similar equation for a spin-5/2 field, defining the Fang-Fronsdal operator $S_{\mu\nu}$,

in analogy with the spin-3/2 case, as

$$S_{\mu\nu} \equiv i\ (\not\partial \psi_{\mu\nu} - \partial_\mu \not\psi_\nu - \partial_\nu \not\psi_\mu)\ =\ 0\ , \tag{47}$$

and generalizing naively the gauge transformation to

$$\delta\psi_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\ . \tag{48}$$

Difficulties similar to those met in the bosonic case for spin 3 readily emerge: this equation is not gauge invariant, but transforms as

$$\delta S_{\mu\nu} = -2i\ \partial_\nu \partial_\mu \not\epsilon\ , \tag{49}$$

and a similar problem will soon arise with the Bianchi identity. However, as in the bosonic case, following Fang and Fronsdal [2], one can constrain both the fermionic field and the gauge parameter ϵ so that the gauge symmetry hold and the Bianchi identity take a non-anomalous form.

We can now consider the generic case of half-integer spin $s + 1/2$ [2]. In the compact notation (33) the equation of motion and the gauge transformation read simply

$$S\ \equiv\ i\,(\not\partial \psi - \partial \not\psi)\ =\ 0\ , \tag{50}$$

$$\delta\psi = \partial\epsilon\ . \tag{51}$$

Since

$$\delta S = -2i\partial^2 \not\epsilon\ , \tag{52}$$

to ensure the gauge invariance of the field equation, one must demand that the gauge parameter be *γ-traceless*,

$$\not\epsilon = 0\ . \tag{53}$$

Let us now turn to the Bianchi identity, computing

$$\partial \cdot S - \frac{1}{2}\partial S' - \frac{1}{2}\not\partial \not S\ , \tag{54}$$

where the last term contains the γ-trace of the operator. It is instructive to consider in detail the individual terms. The trace of S is

$$S' = i(\not\partial \psi' - 2\partial \cdot \not\psi - \partial \not\psi')\ , \tag{55}$$

and therefore, using the rules in (33)

$$-\frac{1}{2}\partial S' = -\frac{i}{2}(\not\partial \partial \psi' - 2\partial\partial \cdot \not\psi - 2\partial^2 \not\psi')\ . \tag{56}$$

Moreover, the divergence of S is

$$\partial \cdot S = i(\not\partial \partial \cdot \psi - \Box \not\psi - \partial\partial \cdot \not\psi)\ . \tag{57}$$

Finally, from the γ trace of S

$$\not S = i(-2\not\partial \not\psi + 2\partial \cdot \psi - \partial \psi')\ , \tag{58}$$

one can obtain

$$-\frac{1}{2}\not\partial \not S = -\frac{i}{2}(-2\Box \not\psi + 2\not\partial \partial \cdot \psi - \partial \not\partial \psi')\ , \tag{59}$$

and putting all these terms together yields the Bianchi identity

$$\partial \cdot S - \frac{1}{2}\partial S' - \frac{1}{2}\not\partial \not S = i\partial^2 \not\psi'\ . \tag{60}$$

As in the bosonic case, this identity contains an "anomalous" term that first manifests itself for spin

$s = 7/2$, and therefore one is lead in general to impose the "triple γ-trace" condition

$$\not\psi' = 0 \ . \tag{61}$$

One can also extend the Kaluza-Klein construction to the spin $s+1/2$ in order to recover the massive Singh-Hagen formulation. The reduction from $D+1$ dimensions to D dimensions will turn the massless field $\psi^{(s)}_{D+1}$ into massive fields of the type $\psi^{(s)}_D, \psi^{(s-1)}_D$ and $\psi^{(s-2)}_D$, while no lower-rank fields can appear because of the triple γ-trace condition (61). In a similar fashion, the gauge parameter $\epsilon^{(s-1)}_{D+1}$ reduces only to a single field $\epsilon^{(s-1)}_D$, as a result of the γ-trace condition (53). Gauge fixing the Stueckelberg symmetries one is finally left with only two fields $\psi^{(s)}_D$ and $\psi^{(s-2)}_D$ that contain precisely the γ-traceless tensors introduced by Singh and Hagen in [9].

3 Free non-local geometric equations

In the previous section we have seen that is possible to construct a Lagrangian for higher-spin bosons imposing the unusual Fronsdal constraints

$$\Lambda' = 0, \qquad \varphi'' = 0 \tag{62}$$

on the fields and on the gauge parameters. Following [3, 4], we can now construct higher-spin gauge theories with unconstrained gauge fields and parameters.

3.1 Non-local Fronsdal-like operators

We can motivate the procedure discussing first in some detail the relatively simple example of a spin-3 field, where

$$\delta \mathcal{F}_{\mu\nu\rho} = 3\, \partial_\mu \partial_\nu \partial_\rho \Lambda' \ . \tag{63}$$

Our purpose is to build a non-local operator $\mathcal{F}_{NL}$ that transforms exactly like the Fronsdal operator $\mathcal{F}$, since the operator $\mathcal{F} - \mathcal{F}_{NL}$ will then be gauge invariant without any additional constraint on the gauge parameter. One can find rather simply the non-local constructs

$$\frac{1}{3\Box}\left[\partial_\mu \partial_\nu \mathcal{F}'_\rho + \partial_\nu \partial_\rho \mathcal{F}'_\mu + \partial_\rho \partial_\mu \mathcal{F}'_\nu\right] , \tag{64}$$

$$\frac{1}{3\Box}\left[\partial_\mu \partial \cdot \mathcal{F}'_{\nu\rho} + \partial_\nu \partial \cdot \mathcal{F}'_{\rho\mu} + \partial_\rho \partial \cdot \mathcal{F}'_{\mu\nu}\right] , \tag{65}$$

$$\frac{1}{\Box^2}\, \partial_\mu \partial_\nu \partial_\rho \partial \cdot \mathcal{F}' \ , \tag{66}$$

but the first two expressions actually coincide, as can be seen from the Bianchi identities (38), and as a result one is led to two apparently distinct non-local fully gauge invariant field equations

$$\mathcal{F}_{\mu_1\mu_2\mu_3} - \frac{1}{3\Box}\left[\partial_{\mu_1} \partial_{\mu_2} \mathcal{F}'_{\mu_3} + \partial_{\mu_2} \partial_{\mu_3} \mathcal{F}'_{\mu_1} + \partial_{\mu_3} \partial_{\mu_1} \mathcal{F}'_{\mu_2}\right] = 0 \ , \tag{67}$$

$$\mathcal{F}^{new}_{\mu_1\mu_2\mu_3} \equiv \mathcal{F}_{\mu_1\mu_2\mu_3} - \frac{1}{\Box^2}\, \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \partial \cdot \mathcal{F}' = 0 \ . \tag{68}$$

These equations can be actually turned into one another, once they are combined with their traces, but the second form, which we denote by $\mathcal{F}^{new}$, is clearly somewhat simpler, since it rests on the addition of the single scalar construct $\partial \cdot \mathcal{F}'$. From $\mathcal{F}^{new}$, one can build in the standard way

$$G_{\mu_1\mu_2\mu_3} \equiv \mathcal{F}^{new}_{\mu_1\mu_2\mu_3} - \frac{1}{2}\left(\eta_{\mu_1\mu_2} \mathcal{F}^{new\,\prime}_{\mu_3} + \eta_{\mu_2\mu_3} \mathcal{F}^{new\,\prime}_{\mu_1} + \eta_{\mu_3\mu_1} \mathcal{F}^{new\,\prime}_{\mu_2}\right) , \tag{69}$$

and one can easily verify that

$$\partial \cdot G_{\mu_1\mu_2} = 0 \ . \tag{70}$$

This identity suffices to ensure the gauge invariance of the Lagrangian. Moreover, we shall see that in all higher-spin cases, the non-local construction will lead to a similar, if more complicated, identity, underlying a Lagrangian formulation that does not need any double trace condition on the gauge field. It is worth stressing this point: we shall see that, modifying the Fronsdal operator in order to achieve gauge invariance without any trace condition on the parameter, the Bianchi identities will change accordingly and the "anomalous" terms will disappear, leading to corresponding gauge invariant Lagrangians.

Returning to the spin-3 case, one can verify that the Einstein tensor $G_{\mu_1\mu_2\mu_3}$ follows from the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}(\partial_\mu \varphi_{\mu_1\mu_2\mu_3})^2 + \frac{3}{2}(\partial\cdot\varphi_{\mu_1\mu_2})^2 - \frac{3}{2}(\partial\cdot\varphi')^2 + \frac{3}{2}(\partial_\mu\varphi'_{\mu_1})^2 \\ & +3\varphi'_{\mu_1}\partial\cdot\partial\cdot\varphi_{\mu_1} + 3\partial\cdot\partial\cdot\partial\cdot\varphi\frac{1}{\Box}\partial\cdot\varphi' - \partial\cdot\partial\cdot\partial\cdot\varphi\frac{1}{\Box^2}\partial\cdot\partial\cdot\partial\cdot\varphi \;, \end{aligned} \tag{71}$$

that is fully gauge invariant under

$$\delta\varphi_{\mu_1\mu_2\mu_3} = \partial_{\mu_1}\Lambda_{\mu_2\mu_3} + \partial_{\mu_2}\Lambda_{\mu_3\mu_1} + \partial_{\mu_3}\Lambda_{\mu_1\mu_2} \;. \tag{72}$$

For all higher spins, one can arrive at the proper analogue of (67) via a sequence of pseudo-differential operators, defined recursively as

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\frac{\partial^2}{\Box}\mathcal{F}^{(n)\prime} - \frac{1}{n+1}\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}^{(n)} \;, \tag{73}$$

where the initial operator $\mathcal{F}^{(1)} = \mathcal{F}$ is the classical Fronsdal operator. The gauge transformations of the $\mathcal{F}^{(n)}$,

$$\delta\mathcal{F}^{(n)} = (2n+1)\frac{\partial^{2n+1}}{\Box^{n-1}}\Lambda^{[n]} \;, \tag{74}$$

involve by construction higher traces of the gauge parameter. Since the n-th trace $\Lambda^{[n]}$ vanishes for all $n > (s-1)/2$, the first corresponding operator $\mathcal{F}^{(n)}$ will be gauge invariant without any constraint on the gauge parameter. A similar inductive argument determines the Bianchi identities for the $\mathcal{F}^{(n)}$,

$$\partial\cdot\mathcal{F}^{(n)} - \frac{1}{2n}\partial\mathcal{F}^{(n)\prime} = -\left(1+\frac{1}{2n}\right)\frac{\partial^{2n+1}}{\Box^{n-1}}\varphi^{[n+1]} \;, \tag{75}$$

where the *anomalous* contribution depends on the $(n+1)$-th trace $\varphi^{[n+1]}$ of the gauge field, and thus vanishes for $n > (s/2-1)$.

The Einstein-like tensor corresponding to $\mathcal{F}^{(n)}$

$$G^{(n)} = \sum_{p=0}^{n-1}\frac{(-1)^p(n-p)!}{2^p n!}\eta^p\mathcal{F}^{(n)[p]} \;, \tag{76}$$

is slightly more complicated than its lower-spin analogs, since it involves in general multiple traces, but an inductive argument shows that

$$\partial\cdot G^{(n)} = 0 \;, \tag{77}$$

so that $G^{(n)}$ follows indeed from a Lagrangian of the type

$$\mathcal{L} \sim \varphi\, G^{(n)} \;. \tag{78}$$

We shall soon see that $G^{(n)}$ has a very neat geometrical meaning. Hence, the field equations, not directly the Lagrangians, are fully geometrical in this formulation.

3.2 Geometric equations

Inspired by General Relativity, we can reformulate the non-local objects like the Ricci tensor $\mathcal{F}^{(n)}$ introduced in the previous section in geometrical terms. We have already seen that, following de Wit and Freedman [14], one can define generalized connections and Riemann tensors of various orders in the derivatives for all spin-s gauge fields as extensions of the spin-2 objects as

$$\begin{array}{rcl} \Gamma_{\mu;\nu_1\nu_2} & \Rightarrow & \Gamma_{\mu_1\ldots\mu_{s-1};\nu_1\ldots\nu_s} \ , \\ R_{\mu_1\mu_2;\nu_1\nu_2} & \Rightarrow & R_{\mu_1\ldots\mu_s;\nu_1\ldots\nu_s} \ , \end{array}$$

and actually a whole hierarchy of connections $\Gamma_{\mu_1\ldots\mu_k;\nu_1\ldots\nu_s}$ whose last two members are the connection and the curvature above. In order to appreciate better the meaning of this generalization, it is convenient to recall some basic facts about linearized Einstein gravity. If the metric is split according to $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the condition that g be covariantly constant leads to the following relation between its deviation h with respect to flat space and the linearized Christoffel symbols:

$$\partial_\rho h_{\mu\nu} = \Gamma_{\nu;\rho\mu} + \Gamma_{\mu;\rho\nu} \ . \tag{79}$$

In strict analogy, the corresponding relation for spin 3 is

$$\partial_\sigma \partial_\tau \varphi_{\mu\nu\rho} \ = \ \Gamma_{\nu\rho;\sigma\tau\mu} + \Gamma_{\rho\mu;\sigma\tau\nu} + \Gamma_{\mu\nu;\sigma\tau\rho} \ . \tag{80}$$

It is possible to give a compact expression for the connections of [14] for arbitrary spin,

$$\Gamma^{(s-1)} = \frac{1}{s} \sum_{k=0}^{s-1} \frac{(-1)^k}{\binom{s-1}{k}} \, \partial^{s-k-1} \nabla^k \varphi \ , \tag{81}$$

where the derivatives ∇ carry indices originating from the gauge field. This tensor is actually the proper analogue of the Christoffel connection for a spin-s gauge field, and transforms as

$$\delta\Gamma_{\alpha_1\cdots\alpha_{s-1};\beta_1\cdots\beta_s} \ \sim \ \partial_{\beta_1} \cdots \partial_{\beta_s} \Lambda_{\alpha_1\cdots\alpha_{s-1}} . \tag{82}$$

That is a direct link between these expressions and the traces of non-local operators of the previous section. From this connection, one can then construct a gauge invariant tensor $\mathcal{R}_{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_s}$ that is the proper analogue of the Riemann tensor of a spin-2 field.

We can thus write in a more compact geometrical form the results of the iterative procedure. The non-local field equations for odd spin $s = 2n+1$ generalizing the Maxwell equations $\partial^\mu F_{\mu;\nu} = 0$ are simply

$$\frac{1}{\Box^n} \partial \cdot \mathcal{R}^{[n]}_{;\mu_1\cdots\mu_{2n+1}} = 0 \ , \tag{83}$$

while the corresponding equations for even spin $s = 2n$ are simply

$$\frac{1}{\Box^{n-1}} \mathcal{R}^{[n]}_{;\mu_1\cdots\mu_{2n}} = 0 \ , \tag{84}$$

that reduce to $R_{;\mu\nu} = 0$ for spin 2.

The non-local geometric equations for higher-spin bosons can be brought to the Fronsdal form using the traces Λ' of the gauge parameters, and propagate the proper number of degrees of freedom. At first sight, however, the resulting Fronsdal equations present a subtlety [4]. The analysis of their physical degrees of freedom normally rests on the choice of de Donder gauge,

$$\mathcal{D} \equiv \partial \cdot \varphi - \frac{1}{2} \, \partial \varphi' = 0 \ , \tag{85}$$

the higher-spin analog of the Lorentz gauge, that reduces the Fronsdal operator to $\Box\varphi$, but *this is a proper gauge only for doubly traceless fields.* The difficulty one faces can be understood noting that,

in order to recover the Fronsdal equation eliminating the non-local terms, one uses the trace Λ' of the gauge parameter. The trace of the de Donder gauge condition, proportional to the double trace φ'' of the gauge field, is then in fact *invariant* under residual gauge transformations with a traceless parameter, so that the de Donder gauge cannot be reached in general. However, it can be modified, as in [4], by the addition of terms containing higher traces of the gauge field, and the resulting gauge fixed equation then sets to the zero the double trace φ'' on shell.

Following similar steps, one can introduce non-local equations for *fermionic* fields with unconstrained gauge fields and gauge parameters [3]. To this end, it is convenient to notice that the fermionic operators for spin $s+1/2$ can be related to the corresponding bosonic operators for spin s according to

$$\mathcal{S}_{s+1/2} - \frac{1}{2}\frac{\partial \not\partial}{\Box} \not\mathcal{S}_{s+1/2} = i\frac{\not\partial}{\Box}\mathcal{F}_s(\psi) \ , \tag{86}$$

that generalize the obvious link between the Dirac and Klein-Gordon operators. For instance, the Rarita-Schwinger equation $\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0$ implies that

$$\mathcal{S} \equiv i(\not\partial\psi_\mu - \partial_\mu \not\psi) = 0 \ , \tag{87}$$

while (86) implies that

$$S_\mu - \frac{1}{2}\frac{\partial_\mu \not\partial}{\Box}\not\mathcal{S} \ = \ \frac{i\not\partial}{\Box}[\eta_{\mu\nu} - \partial_\mu\partial_\nu]\psi^\nu \ . \tag{88}$$

Non-local fermionic kinetic operators $\mathcal{S}^{(n)}$ can be defined recursively as

$$\mathcal{S}^{(n+1)} = \mathcal{S}^{(n)} + \frac{1}{n(2n+1)}\frac{\partial^2}{\Box}\mathcal{S}^{(n)\prime} - \frac{2}{2n+1}\frac{\partial}{\Box}\partial\cdot\mathcal{S}^{(n)} \ , \tag{89}$$

with the understanding that, as in the bosonic case, the iteration procedure stops when the gauge variation

$$\delta\mathcal{S}^{(n)} = -2i\, n\,\frac{2n}{\Box^{n-1}}\not\epsilon^{\,[n-1]} \tag{90}$$

vanishes due to the impossibility of constructing the corresponding higher trace of the gauge parameter. The key fact shown in [3,4] is that, as in the bosonic case, the Bianchi identities are similarly modified, according to

$$\partial\mathcal{S}^{(n)} \ - \ \frac{1}{2n}\,\partial\,\mathcal{S}^{(n)\,\prime} \ - \ \frac{1}{2n}\,\not\partial\not\mathcal{S}^{(n)} \ = \ i\,\frac{\partial^{\,2n}}{\Box^{\,n-1}}\not\psi^{[n]} \ , \tag{91}$$

and lack the anomalous terms when n is large enough to ensure that the field equations are fully gauge invariant. Einstein-like operators and field equations can then be defined following steps similar to those illustrated for the bosonic case.

4 Triplets and local compensator form

4.1 String field theory and BRST

String Theory includes infinitely many higher-spin massive fields with consistent mutual interactions, and it tensionless limit $\alpha' \to \infty$ lends itself naturally to provide a closer view of higher-spin fields. Conversely, a better grasp of higher-spin dynamics is likely to help forward our current understanding of String Theory.

Let us recall some standard properties of the open bosonic string oscillators. In the mostly plus convention for the metric, their commutations relations read

$$[\alpha_k^\mu, \alpha_l^\nu] = k\delta_{k+l,0}\eta^{\mu\nu} \ , \tag{92}$$

and the corresponding Virasoro operators

$$L_k = \frac{1}{2}\Sigma_{l=-\infty}^{+\infty}\alpha_{k-l}^{\mu}\alpha_{\mu l}, \tag{93}$$

where $\alpha_0^{\mu} = \sqrt{2\alpha'}p^{\mu}$ and $p_{\mu} - i\partial_{\mu}$ satisfy the Virasoro algebra

$$[L_k, L_l] = (k-l)L_{k+l} + \frac{\mathcal{D}}{12}m(m^2-1) \ , \tag{94}$$

where the central charge equals the space-time dimension $\mathcal{D}$.

In order to study the tensionless limit, it is convenient to rescale the Virasoro generators according to

$$L_k \to \frac{1}{\sqrt{2\alpha'}}L_k, \qquad L_0 \to \frac{1}{\alpha'}L_0. \tag{95}$$

Taking the limit $\alpha' \to \infty$, one can then define the reduced generators

$$l_0 = p^2, \quad l_m = p\cdot\alpha_m \qquad (m \neq 0) \ , \tag{96}$$

that satisfy the simpler algebra

$$[l_k, l_l] = k\delta_{k+l,0}l_0 \ . \tag{97}$$

Since this contracted algebra does not contain a central charge, the resulting massless models are consistent in any space-time dimension, in sharp contrast with what happens in String Theory when α' is finite. It is instructive to compare the mechanism of mass generation at work in String Theory with the Kaluza-Klein reduction, that as we have seen in previous sections works for arbitrary dimensions. A closer look at the first few mass levels shows that, *as compared to the Kaluza-Klein setting, the string spectrum lacks some auxiliary fields, and this feature may be held responsible for the emergence of the critical dimension!*

Following the general BRST method, let us introduce the ghost modes C_k of ghost number one and the corresponding antighosts B_k of ghost number minus one, with the usual anti-commutation relations. The BRST operator [17, 18, 19]

$$\mathcal{Q} = \sum_{-\infty}^{+\infty}[C_{-k}L_k - \frac{1}{2}(k-l) \ : \ C_{-k}C_{-l}B_{k+l} \ :] - C_0 \tag{98}$$

determines the free string equation

$$\mathcal{Q}|\Phi\rangle = 0 \ , \tag{99}$$

while the corresponding gauge transformation is

$$\delta|\Phi\rangle = \mathcal{Q}|\Lambda\rangle \ . \tag{100}$$

Rescaling the ghost variables according to

$$c_k = \sqrt{2\alpha'}C_k \ , \quad b_k = \frac{1}{2\alpha'}B_k \ , \tag{101}$$

for $k \neq 0$ and as

$$c_0 = \alpha' C_0, \quad b_0 = \frac{1}{\alpha'}B_0 \tag{102}$$

for $k = 0$ allows a non-singular $\alpha' \to \infty$ limit that defines the *identically* nilpotent BRST charge

$$Q = \sum_{-\infty}^{+\infty}[c_{-k}l_k - \frac{k}{2}b_0c_{-k}c_k] \ . \tag{103}$$

Making the zero-mode structure manifest then gives

$$Q = c_0 l_0 - b_0 M + \tilde{Q} \ , \tag{104}$$

where $\tilde{Q} = \sum_{k\neq 0} c_{-k} l_k$ and $M = \frac{1}{2}\sum_{-\infty}^{+\infty} k c_{-k} c_k$, and the string field and the gauge parameter can be decomposed as

$$\begin{aligned} |\Phi\rangle &= |\varphi_1\rangle + c_0|\varphi_2\rangle \ , & (105)\\ |\Lambda\rangle &= |\Lambda_1\rangle + c_0|\Lambda_2\rangle \ . & (106)\end{aligned}$$

It should be appreciated that *in this formulation no trace constraint is imposed on the master gauge field φ or on the master gauge parameter* Λ. It is simple to confine the attention to totally symmetric tensors, selecting states $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\Lambda\rangle$ that are built from a single string mode α_{-1},

$$\begin{aligned} |\varphi_1\rangle &= \sum_{s=0}^{\infty} \frac{1}{s!}\,\varphi_{\mu_1\cdots\mu_s}(x)\alpha_{-1}^{\mu_1}\cdots\alpha_{-1}^{\mu_s}|0\rangle \ , \\ &+ \sum_{s=2}^{\infty}\frac{1}{(s-2)!}D_{\mu_1\cdots\mu_{s-2}}(x)\alpha_{-1}^{\mu_1}\cdots\alpha_{-1}^{\mu_{s-2}}c_{-1}b_{-1}|0\rangle \ , & (107)\\ |\varphi_2\rangle &= \sum_{s=1}^{\infty}\frac{-i}{(s-1)!}C_{\mu_1\cdots\mu_{s-1}}(x)\alpha_{-1}^{\mu_1}\cdots\alpha_{-1}^{\mu_{s-1}}b_{-1}|0\rangle \ , & (108)\\ |\Lambda\rangle &= \sum_{s=1}^{\infty}\frac{i}{(s-1)!}\Lambda_{\mu_1\cdots\mu_{s-1}}(x)\alpha_{-1}^{\mu_1}\cdots\alpha_{-1}^{\mu_{s-1}}b_{-1}|0\rangle \ . & (109)\end{aligned}$$

Restricting eqs. (99) and (100) to states of this type, the s-th terms of the sums above yield the *triplet* equations [20, 21, 4]

$$\begin{aligned} &\Box\varphi = \partial C \ ,\\ &\partial\cdot\varphi - \partial D = C \ ,\\ &\Box D = \partial\cdot C \ ,\end{aligned} \tag{110}$$

and the corresponding gauge transformations

$$\begin{aligned} \delta\varphi &= \partial\Lambda \ ,\\ \delta C &= \Box\Lambda \ ,\\ \delta D &= \partial\cdot\Lambda \ ,\end{aligned} \tag{111}$$

where φ is rank-s tensor, C is a rank-$(s-1)$ tensor and D is a rank-$(s-2)$ tensor. These field equations follow from a corresponding truncation of the Lagrangian

$$\mathcal{L} = \langle\Phi|Q|\Phi\rangle \ , \tag{112}$$

that in component notation reads

$$\mathcal{L} \ = \ -\frac{1}{2}(\partial_\mu\varphi)^2 + s\partial\cdot\varphi C + s(s-1)\partial\cdot C\, D + \frac{s(s-1)}{2}(\partial_\mu D)^2 - \frac{s}{2}C^2 \ , \tag{113}$$

where the D field, whose modes disappear on the mass shell, has a peculiar negative kinetic term. Note

that one can also eliminate the auxiliary field C, thus arriving at the equivalent formulation

$$\begin{aligned}\mathcal{L} &= -\sum_{s=0}^{+\infty} \frac{1}{s!}\big[\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{s}{2}(\partial\cdot\varphi)^2 + s(s-1)\partial\cdot\partial\cdot\varphi\, D \\ &+ s(s-1)(\partial_\mu D)^2 + \frac{s(s-1)(s-2)}{2}(\partial\cdot D)^2\big] \ . \qquad (114)\end{aligned}$$

in terms of pairs (φ, D) of symmetric tensors, more in the spirit of [20].

For a given value of s, this system propagates modes of spin s, $s-2$, ..., down to 0 or 1 according to whether s is even or odd. This can be simply foreseen from the light-cone description of the string spectrum, since the corresponding physical states are built from arbitrary powers of a single light-cone oscillator α_1^i, that taking out traces produces precisely a nested chain of states with spins separated by two units. From this reducible representation, as we shall see, it is possible to deduce a set of equations for an irreducible multiplet demanding that the trace of φ be related to D.

If the auxiliary C field is eliminated, the equations of motion for the triplet take the form

$$\mathcal{F} = \partial^2(\varphi' - 2D) \ , \qquad (115)$$

$$\Box D = \frac{1}{2}\partial\cdot\partial\cdot\varphi - \frac{1}{2}\partial\partial\cdot D \ . \qquad (116)$$

4.2 (A)dS extensions of the bosonic triplets

The interaction between a spin 3/2 field and the gravitation field is described essentially by a Rarita-Schwinger equation where ordinary derivatives are replaced by Lorentz-covariant derivatives. The gauge transformation of the Rarita-Schwinger Lagrangian is then surprisingly proportional not to the Riemann tensor, but to the Einstein tensor, and this variation is precisely compensated in supergravity [10] by the supersymmetry variation of the Einstein Lagrangian. However, if one tries to generalize this result to spin $s \geq 5/2$, the miracle does not repeat and the gauge transformations of the field equations of motion generate terms proportional to the Riemann tensor, and similar problems are also met in the bosonic case. This is the Aragone-Deser problem for higher spins [22].

As was first noticed by Fradkin and Vasiliev [23], with a non-vanishing cosmological constant Λ it is actually possible to modify the spin $s \geq 5/2$ field equations introducing additional terms that depend on negative powers of Λ and cancel the dangerous Riemann curvature terms. This observation plays a crucial role in the Vasiliev equations [24], discussed in the lectures by Vasiliev [25] and Sundell [26] at this Workshop.

For these reasons it is interesting to describe the (A)dS extensions of the massless triplets that emerge from the bosonic string in the tensionless limit and the corresponding deformations of the compensator equations. Higher-spin gauge fields propagate consistently and independently of one another in conformally flat space-times, bypassing the Aragone-Deser inconsistencies that would be introduced by a background Weyl tensor, and this free-field formulation in an (A)dS background serves as a starting point for exhibiting the unconstrained gauge symmetry of [3,4], as opposed to the constrained Fronsdal gauge symmetry [1], in the recent form of the Vasiliev equations [24] based on vector oscillators [26].

One can build the (A)dS symmetric triplets from a modified BRST formalism [5], but in the following we shall rather build them directly deforming the flat triplets. The gauge transformations of φ and D are naturally turned into their curved-space counterparts,

$$\delta\varphi = \nabla\Lambda \ , \qquad (117)$$

$$\delta D = \nabla\cdot\Lambda \ , \qquad (118)$$

where the commutator of two covariant derivatives on a vector in AdS is

$$[\nabla_\mu, \nabla_\nu]V_\rho = \frac{1}{L^2}(g_{\nu\rho}V_\mu - g_{\mu\rho}V_\nu) \ . \qquad (119)$$

However, in order to maintain the definition of $C = \nabla\cdot\varphi - \nabla D$, one is led to deform its gauge variation,

turning it into

$$\delta C = \Box\Lambda + \frac{(s-1)(3-s-\mathcal{D})}{L^2}\Lambda + \frac{2}{L^2} g\Lambda' \ , \tag{120}$$

where $-1/L^2$ is the AdS cosmological constant and g is the background metric tensor. The corresponding de Sitter equations could be obtained by the formal continuation of L to imaginary values, $L \to iL$.

These gauge transformations suffice to fix the other equations, that read

$$\Box\varphi = \nabla C - \frac{1}{L^2}\left\{-8gD + 2g\varphi' - [(2-s)(3-\mathcal{D}-s) - s]\varphi\right\} \ , \tag{121}$$

$$C = \nabla\cdot\varphi - \nabla D, \tag{122}$$

$$\Box D = \nabla\cdot C - \frac{1}{L^2}\left\{-[s(\mathcal{D}+s-2)+6]D + 4\varphi' + 2gD'\right\} \ , \tag{123}$$

and as in the previous section one can also eliminate C. To this end, it is convenient to define the AdS Fronsdal operator, that extends (34), as

$$\mathcal{F} \equiv \Box\varphi - \nabla\nabla\cdot\varphi + \frac{1}{2}\{\nabla,\nabla\}\varphi' \ . \tag{124}$$

The first equation of (121) then becomes

$$\mathcal{F} \quad = \quad \frac{1}{2}\{\nabla,\nabla\}\left(\varphi' - 2D\right) + \frac{1}{L^2}\left\{8gD - 2g\varphi' + [(2-s)(3-\mathcal{D}-s) - s]\varphi\right\} \ . \tag{125}$$

In a similar fashion, after eliminating the auxiliary field C, the AdS equation for D becomes

$$\begin{aligned}\Box D + \frac{1}{2}\nabla\nabla\cdot D \quad &- \quad \frac{1}{2}\nabla\cdot\nabla\cdot\varphi = -\frac{(s-2)(4-\mathcal{D}-s)}{2L^2}D - \frac{1}{L^2}gD' \\ &+ \quad \frac{1}{2L^2}\left\{[s(\mathcal{D}+s-2)+6]D - 4\varphi' - 2gD'\right\} \ . \end{aligned}\tag{126}$$

It is also convenient to define the modified Fronsdal operator

$$\mathcal{F}_L = \mathcal{F} - \frac{1}{L^2}\left\{[(3-\mathcal{D}-s)(2-s) - s]\varphi + 2g\varphi'\right\} \ , \tag{127}$$

since in terms of $\mathcal{F}_L$ eq. (125) becomes

$$\mathcal{F}_L = \frac{1}{2}\{\nabla,\nabla\}(\varphi' - 2D) + \frac{8}{L^2}\, g\, D \ , \tag{128}$$

while the Bianchi identity becomes simply

$$\nabla\cdot\mathcal{F}_L - \frac{1}{2}\nabla\mathcal{F}_L' = -\frac{3}{2}\nabla^3\varphi'' + \frac{2}{L^2} g\,\nabla\varphi'' \ . \tag{129}$$

4.3 Compensator form of the bosonic equations

In the previous sections we have displayed a non-local geometric Lagrangian formulation for higher-spin bosons and fermions. In this section we show how one can obtain very simple *local non-Lagrangian* descriptions that exhibit the unconstrained gauge symmetry present in the non-local equations and reduce to the Fronsdal form after a partial gauge fixing [3, 4, 5].

The key observation is that the case of a single propagating spin-s field can be recovered from the equations (115)-(116) demanding that all lower-spin excitations be pure gauge. To this end, it suffices to introduce a spin $s-3$ *compensator* α as

$$\varphi' - 2D = \partial\alpha \ , \tag{130}$$

that by consistency transforms as

$$\delta\alpha = \Lambda' \ . \tag{131}$$

Eq. (115) then becomes

$$\mathcal{F} = 3\partial^3\alpha \ , \tag{132}$$

while (116) becomes

$$\mathcal{F}' - \partial^2\varphi'' = 3\,\Box\partial\alpha \ + \ 2\,\partial^2\partial\cdot\alpha \ . \tag{133}$$

Combining them leads to

$$\partial^2\varphi'' \ = \ \partial^2(4\partial\cdot\alpha + \partial\alpha') \ , \tag{134}$$

and the conclusion is then that the triplet equations imply a pair of *local* equations for a single massless spin-s gauge field φ and a single spin-$(s-3)$ compensator α. Summarizing, the local compensator equations and the corresponding gauge transformations are

$$\begin{aligned} &\mathcal{F} = 3\partial^3\alpha, \quad \varphi'' = 4\partial\cdot\alpha + \partial\alpha' \ , \\ &\delta\varphi = \partial\Lambda, \qquad \delta\alpha = \Lambda' \ , \end{aligned} \tag{135}$$

and clearly reduce to the standard Fronsdal form after a partial gauge fixing using the trace Λ' of the gauge parameter . These equations can be regarded as the local analogs of the non-local geometric equations, but it should be stressed that *they are not Lagrangian equations*. This can be seen either directly, as in [4, 5], or via the corresponding BRST operator, that is not hermitian, as pertains to a reduced system that is not described by a Lagrangian [27]. Nonetheless, the two equations (135) form a consistent system, and the first can be turned into the second using the Bianchi identity.

One can also obtain the (A)dS extension of the spin-s compensator equations (135). The natural starting point are the (A)dS gauge transformations for the fields φ and α

$$\delta\varphi = \nabla\Lambda \ , \qquad \delta\alpha = \Lambda' \ . \tag{136}$$

One can then proceed in various ways to obtain the compensator equations

$$\begin{aligned} \mathcal{F} \ &= \ 3\nabla^3\alpha + \frac{1}{L^2}\left\{-2g\varphi' + [(2-s)(3-\mathcal{D}-s)-s]\varphi\right\} - \frac{4}{L^2}g\nabla\alpha \\ \varphi'' \ &= \ 4\partial\cdot\alpha + \partial\alpha' \ , \end{aligned} \tag{137}$$

that, of course, again do *not* follow from a Lagrangian. However, lagrangian equations can be obtained, both in flat space and in an (A)dS background, from a BRST construction based on a wider set of constraints first obtained by Pashnev and Tsulaia [28, 5]. It is instructive to illustrate these results for spin 3 bosons.

In addition to the triplet fields φ, C and D and the compensator α, this formulation uses the additional spin-1 fields $\varphi^{(1)}$ and F and spin-0 fields $C^{(1)}$ and E, together with a new spin-1 gauge parameter μ and a new spin-0 gauge parameter $\Lambda^{(1)}$. The BRST analysis generates the gauge transformations

$$\begin{aligned} &\delta\varphi = \partial\Lambda + \eta\,\mu \ , && \delta\alpha = \Lambda' - \sqrt{2\mathcal{D}}\Lambda^{(1)} \ , \\ &\delta\varphi^{(1)} = \partial\Lambda^{(1)} + \sqrt{\tfrac{\mathcal{D}}{2}}\mu \ , && \delta C = \Box\Lambda \ , \\ &\delta D = \partial\cdot\Lambda + \mu \ , && \delta C^{(1)} = \Box\Lambda^{(1)} \ , \\ &\delta E = \partial\cdot\mu \ , && \delta F = \Box\mu \ , \end{aligned} \tag{138}$$

and the corresponding field equations

$$\begin{aligned} &\Box\varphi = \partial C + \eta F \ , && \Box\alpha C' - \sqrt{2\mathcal{D}}C^{(1)} \ , \\ &\partial\cdot\varphi - \partial D - \eta E = C \ , && \Box\varphi^{(1)} = \partial C^{(1)} + \sqrt{\tfrac{\mathcal{D}}{2}}F \ , \\ &\Box D = \partial\cdot C + F \ , && \Box E = \partial\cdot F \ , \\ &\partial\alpha = \varphi' - 2D - \sqrt{2\mathcal{D}}\varphi^{(1)} \ , && \partial\cdot\varphi^{(1)} - \sqrt{2\mathcal{D}}E = C^{(1)} \ . \end{aligned} \tag{139}$$

Making use of the gauge parameters μ and $\Lambda^{(1)}$ one can set $\varphi^{(1)} = 0$ and $C^{(1)} = 0$, while the other additional fields are set to zero by the field equations. Therefore, one can indeed recover the non-Lagrangian compensator equations gauge fixing this Lagrangian system. A similar, if more complicated analysis, goes through for higher spins, where this formulation requires $\mathcal{O}(s)$ fields.

The logic behind these equations can be captured rather simply taking a closer look at the gauge transformations (138). One is in fact gauging away φ', modifying the gauge transformation of φ by the μ term. This introduces a corresponding modification in the φ equation, that carries through by integrability to the C equation, and so on.

4.4 Fermionic triplets

We can now turn to the fermionic triplets proposed in [4] as a natural guess for the field equations of symmetric spinor-tensors arising in the tensionless limit of superstring theories. In fact, the GSO projection limits their direct occurrence to type-0 theories [29], but slightly more complicated spinor-tensors of this type, but with mixed symmetry, are present in all superstring spectra, and can be discussed along similar lines [5].

The counterparts of the bosonic triplet equations and gauge transformations are

$$\begin{array}{ll} \not\partial\psi = \partial\chi \ , & \delta\psi = \partial\epsilon \ , \\ \partial\cdot\psi - \partial\lambda = \not\partial\chi \ , & \delta\lambda = \partial\cdot\epsilon \ , \\ \not\partial\lambda = \partial\cdot\chi \ , & \delta\chi = \not\partial\epsilon \ . \end{array} \tag{140}$$

It can be shown that this type of system propagates spin-$(s+1/2)$ modes and *all* lower half-integer spins. One can now introduce a spin-$(s-2)$ compensator ξ proceeding in a way similar to what we have seen for the bosonic case, and the end result is a simple non-Lagrangian formulation for a single spin-s field,

$$\begin{array}{ll} \mathcal{S} = -2\, i\, \partial^2\xi \ , & \delta\psi = \partial\epsilon \ , \\ \not\psi' = 2\,\partial\cdot\xi + \partial\xi' + \not\partial\not\xi \ , & \delta\xi = \not\epsilon \ . \end{array} \tag{141}$$

These equations turn into one another using the Bianchi identity, and can be extended to (A)dS background, as in [5]. However, a difficulty presents itself when attempting to extend the fermionic triplets to off-shell systems in (A)dS, since the BRST analysis shows that the operator extension does not define a closed algebra.

4.5 Fermionic compensators

One can also extend nicely the fermionic compensator equations to an (A)dS background. The gauge transformation for a spin-$(s+1/2)$ fermion is deformed in a way that can be anticipated from supergravity and becomes in this case

$$\delta\psi = \nabla\epsilon + \frac{1}{2L}\gamma\epsilon \ , \tag{142}$$

where L determines again the (A)dS curvature and ∇ denotes an (A)dS covariant derivative. The commutator of two of these derivatives on a spin-1/2 field η reads

$$[\nabla_\mu, \nabla_\nu]\eta = -\frac{1}{2L^2}\,\gamma_{\mu\nu}\eta \ , \tag{143}$$

and using eqs (119)-(143) one can show that the compensator equations for a spin-s fermion ($s = n+1/2$) in an (A)dS background are

$$
\begin{aligned}
(\not\nabla \psi - \nabla \not\psi) \quad &+ \quad \frac{1}{2L}[\mathcal{D} + 2(n-2)]\psi + \frac{1}{2L}\gamma\not\psi \\
&= \quad -\{\nabla, \nabla\}\xi + \frac{1}{L}\gamma\nabla\xi + \frac{3}{2L^2}g\xi \ , \\
\not\psi' \quad &= \quad 2\nabla\cdot\xi + \not\nabla\not\xi + \nabla\xi' + \frac{1}{2L}[\mathcal{D} + 2(n-2)]\not\xi - \frac{1}{2L}\gamma\xi' .
\end{aligned}
\tag{144}
$$

These equations are invariant under

$$\delta\psi \quad = \quad \nabla\epsilon \ , \tag{145}$$

$$\delta\xi \quad = \quad \not\epsilon \ , \tag{146}$$

with an unconstrained parameter ϵ. Eqs. (144) are again a pair of non-lagrangian equations, like their flat space counterparts (141).

As in the flat case, eqs (144) are nicely consistent, as can be shown making use of the (A)dS deformed Bianchi identity (60)

$$\nabla\cdot\mathcal{S} - \frac{1}{2}\nabla\mathcal{S}' - \frac{1}{2}\not\nabla\not\mathcal{S} \quad = \quad \frac{i}{4L}\gamma\mathcal{S}' + \frac{i}{4L}[(\mathcal{D}-2) + 2(n-1)]\not\mathcal{S} \tag{147}$$

$$+ \quad \frac{i}{2}\left[\{\nabla, \nabla\} - \frac{1}{L}\gamma\nabla - \frac{3}{2L^2}\right]\not\psi' \ , \tag{148}$$

where the Fang-Fronsdal operator $\mathcal{S}$ is also deformed and becomes

$$\mathcal{S} = i(\not\nabla\psi - \nabla\not\psi) + \frac{i}{2L}[\mathcal{D} + 2(n-2)]\psi + \frac{i}{2L}\gamma\not\psi \ . \tag{149}$$

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On Higher Spins with a Strong $Sp(2,R)$ Condition

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Abstract. We report on an analysis of the Vasiliev construction for minimal bosonic higher-spin master fields with oscillators that are vectors of $SO(D-1,2)$ and doublets of $Sp(2,R)$. We show that, if the original master field equations are supplemented with a strong $Sp(2,R)$ projection of the 0-form while letting the 1-form adjust to the resulting Weyl curvatures, the linearized on-shell constraints exhibit both the proper mass terms and a geometric gauge symmetry with unconstrained, traceful parameters. We also address some of the subtleties related to the strong projection and the prospects for obtaining a finite curvature expansion.

Based in part on the lectures presented by P. Sundell at the "First Solvay Workshop on Higher-Spin Gauge Theory", Brussels, May 12-14 2004, on the lectures presented by A. Sagnotti at "Modern Trends in String Theory, II", Oporto, June 21-26 2004, on the talk presented by A. Sagnotti at the "37th Symposium Ahrenshoop", Berlin-Schmoeckwitz, August 23-27 2004, and on the talks presented by E. Sezgin at the "Third International Symposium on Quantum Theory and Symmetries", (QTS3), Cincinnati, Ohio, USA, September 10 - 14, 2003 and at "Deserfest & The Status of M-Theory", Ann Arbor, Michigan, USA, 3-7 April 2004

1 Introduction

A consistent interacting higher-spin (HS) gauge theory can be viewed as a generalization of Einstein gravity by the inclusion of infinitely many massless HS fields, together with finitely many lower-spin fields including one or more scalars [1]. There are many reasons, if not a rigorous proof, that point to the inevitable presence of infinite towers of fields in these systems.

Although HS gauge theories were studied for a long time for their own sake [1] (see, for instance, [3,4,5] for reviews and more references to the original literature), the well-known recent advances on holography in AdS have shifted to some extent the emphasis toward seeking connections with String/M Theory. Indeed, tensionless strings should lie behind massless higher spins in AdS, as proposed in [6], with the leading Regge trajectory becoming massless at some critical value of the tension [7], and interesting work in support of the latter proposal was recently done in [8]. The theory of massless HS fields has a long history, and the literature has been growing considerably in recent years. Therefore, rather than attempting to review the subject, we shall stress that consistent, fully interacting *and* supersymmetric HS gauge theories have not been constructed yet in dimensions beyond four (although the linearized field equations [9] and certain cubic couplings [10] are known in 5D, and to a lesser extent in 7D [7]). This was also the case for bosonic HS gauge theories, until Vasiliev [11] recently proposed a set of nonlinear field equations for totally symmetric tensors in arbitrary dimensions. The new ingredient is the use of an $Sp(2,R)$ doublet of oscillators valued in the vector representation of $SO(D-1,2)$ [12], called Y^{iA} in the following, as opposed to the bosonic spinor oscillators [13] used in the original construction of the 4D theory [1].

The two formulations[2] rest on the same sequence of minimal HS Lie algebra extensions of $SO(D-1,2)$ [34,15]. Not surprisingly, the *associative* nature of the oscillator $\star$-product algebra, discussed in Section 3.2, is reflected in the possibility of extending the HS field equations to matrix-valued master fields associated to the classical algebras, although here we shall actually confine our attention to the minimal HS gauge theory, where the matrices are one-dimensional. We would like to emphasize, however, that *the matrix extensions have the flavor of the Chan-Paton generalizations of open strings [14], and their consistency actually rests of the same key features of the matrix algebras, which suggests a natural link of the Vasiliev equations to open, rather than to closed strings, in the tensionless limit.* More support for this view will be presented in [15].

The Vasiliev construction [11] requires, as in [1], a pair of master fields, a 1-form $\widehat{A}$ and a 0-form $\widehat{\Phi}$. It rests on a set of integrable constraints on the exterior derivatives $d\widehat{A}$ and $d\widehat{\Phi}$ that, when combined with Lorentz-trace conditions on the component fields, give the equations of motion within the framework of unfolded dynamics [16], whereby a given field is described via an infinity of distinct, albeit related, tensor fields. Notice that here *all* curvatures are constrained, so that the condition on the 2-form curvature embodies generalized torsion constraints and identifies generalized Riemann curvatures with certain components of $\widehat{\Phi}$, while the constraint on the 1-form curvature subsumes the Bianchi identities as well as the very definition of an unfolded scalar field. The presence of trace parts allows in this vector construction both an "off-shell" formulation, that defines the curvatures, and an "on-shell" formulation, where suitable trace conditions turn these definitions into dynamical equations. This is in contrast with the original 4D spinor construction [1], that is only in "on-shell" form, since the special properties of $SL(2,C)$ multi-spinors make its 0-form components inevitably traceless.

Aside from leading to an expansion of the 0-form master field in terms of traceful, and thus off-shell, Riemann tensors, the vector oscillators also introduce an $Sp(2,R)$ redundancy. The crucial issue is therefore how to incorporate the trace conditions for on-shell Riemann tensors and scalar-field derivatives into suitable $Sp(2,R)$-invariance conditions on the master fields. Here, and in a more extensive paper that we hope to complete soon [17], we would like to propose that the Vasiliev equations of [11] be

[1] A notable exception is [2], where a connection between massless HS gauge theories and the eleven-dimensional supermembrane was proposed.

[2] The precise relation between the vector and spinor-oscillator formulations of the Vasiliev equations is yet to be determined.

supplemented with the *strong* $Sp(2,R)$-invariance condition

$$\widehat{K}_{ij} \ \star\ \widehat{\Phi} \ = \ 0 \ , \tag{1}$$

where the $\widehat{K}_{ij}$ are fully interacting $Sp(2,R)$ generators, while subjecting $\widehat{A}$ only to the *weak* condition

$$\widehat{D}\widehat{K}_{ij} \ = \ 0 \ . \tag{2}$$

Whereas one could consider alternative forms of the strong projection condition (1), involving various higher-order constructs of the $Sp(2,R)$ generators, the above linear form is motivated to some extent by the Lagrangian formulation of tensionless strings, or rather, string bits [15], as well as by related previous works of Bars and others on two-time physics and the use of Moyal products in String Field Theory [12, 18]. There is one crucial difference, however, in that the Vasiliev equations (1) involve an additional set of vector oscillators, called Z^{iA} in the following, whose origin in the tensionless limit will be made more transparent in [15]. Further support for (1) is provided by previous constructions of linearized 5D and 7D HS gauge theories based on commuting spinor oscillators [4, 19], where the linearized trace conditions are naturally incorporated into strong $U(1)$ and $SU(2)$ projections of the corresponding Weyl 0-forms. As we shall see, the projection (1) imposes trace conditions leading to correct mass-terms in the linearized field equations for the Weyl tensors and the scalar field, while the resulting gauge field equations exhibit a mixing phenomenon. To wit, the Einstein metric arises in an admixture with the scalar field, that can be resolved by a Weyl rescaling.

The strong $Sp(2,R)$ projection (1) introduces non-polynomial redefinitions of the linearized zero-form Φ via "dressing functions", similar to those that first appeared in the spinorial constructions of [4, 19], corresponding to a projector M, via $\Phi = M\star C$ as discussed in Section 4.3. It turns out that this projector is singular in the sense that $M\star M$ is divergent, and thus M is not normalizable [20, 34]. This poses a potential obstruction to a well-defined curvature expansion of the Vasiliev equations based on the vector oscillators. We have started to address this issue, and drawing on the similar, if simpler, $U(1)$ projection of [4, 20], we discuss how, in principle, one could extract nonetheless a finite curvature expansion. At the present time, however, it cannot be fully excluded that the vector-oscillator formulation of [11] contains a pathology, so that consistent interactions would only exist in certain low dimensions fixed by the HS algebra isomorphisms to spinor-oscillator realizations outlined in Section 2, although this is unlikely. In fact, the presence of such limitations would be in line with the suggestion that the Vasiliev equations be somehow related to String Field Theory, and might therefore retain the notion of a critical dimension, but no such constraints are visible in the tensionless limit of free String Field Theory, that rests on a contracted form of the Virasoro algebra where the central charge has disappeared altogether. A final word on the matter would require a more thorough investigation of the actual interactions present in the $Sp(2,R)$ system, to which we plan to return soon [17].

An additional result discussed here concerns the existence of a direct link between the linearized equations and the geometric formulation with *traceful* gauge fields and parameters of [21, 22]. The strong condition (1) on $\widehat{\Phi}$, together with the weak condition (2) on $\widehat{A}$, implies indeed that the spin-s gauge field equations embodied in the Vasiliev equations take the local compensator form

$$\mathcal{F}_{a_1\dots a_s} = \nabla_{(a_1}\nabla_{a_2}\nabla_{a_3}\alpha_{a_4\dots a_s)} + \text{AdS covariantizations} \ , \tag{3}$$

where $\mathcal{F}$ denotes the Fronsdal operator [23] and α is a spin-$(s-3)$ compensator, that here often differs from its definition in [21, 22] by an overall normalization. This result reflects the link, first discussed by Bekaert and Boulanger [24], between the Freedman-de Wit connections [25] and the compensator equations of [21, 22]. These non-Lagrangian compensator equations are equivalent, in their turn, to the non-local geometric equations of [21], where the Fronsdal operator is replaced by a higher-spin curvature.

By and large, we believe that a thorough understanding of HS gauge theories can be instrumental in approaching String Theory at a deeper conceptual level, since the HS symmetry already implies a far-reaching extension of the familiar notion of spacetime. In particular, the unfolded formulation [16] embodies an unusually large extension of diffeomorphism invariance, in that it is intrinsically independent of the notion of spacetime coordinates, and therefore lends itself to provide a natural geometric basis for

translation-like gauge symmetries, somehow in the spirit of the discussion of supertranslations in [26].

The plan of this article is as follows. We begin in Section 2 by relating the HS symmetries to tensionless limits of strings and branes in AdS, and continue in Section 3 with the off-shell definitions of the curvatures, to move on, in Section 4, to the on-shell theory and the linearized field equations, including their compensator form. Finally, in Section 5 we examine the singular projector and other related non-polynomial objects, primarily for the simpler 5D $U(1)$ case, and suggest how one could arrive at a finite curvature expansion.

This article covers part of the lectures delivered by one of us (P.S.) at the First Solvay Workshop on Higher-Spin Gauge Theories. A group of students was expected to edit the complete lectures and to co-author the resulting manuscript with the speaker, as was the case for other Solvay talks, but this arrangement turned out not to be possible in this case. We thus decided to write together this contribution, combining it with remarks made by the other authors at other Meetings and with some more recent findings, in order to make some of the results that were presented in Brussels available. Hopefully, we shall soon discuss extensively these results in a more complete paper [17], but a fully satisfactory analysis cannot forego the need for a better grasp of the non-linear interactions in the actual $SP(2,R)$ setting. The remaining part of the lectures of P.S. was devoted to the relation between bosonic string bits in the low-tension limit, minimal HS algebras and master equations. Their content will be briefly mentioned in the next section, but will be published elsewhere [15], together with the more recent results referred to above, that were obtained in collaboration with J. Engquist.

2 Higher spins and tensionless limits

HS symmetries naturally arise in the tensionless limits of strings and branes in AdS, and there is a growing literature on this very interesting subject, for instance [27, 28, 29, 12, 6, 18, 7, 30, 22, 31, 32, 8, 33]. The weakly coupled description of a p-brane with small tension is a system consisting of discrete degrees of freedom, that can be referred to as “bits”, each carrying finite energy and momentum [29]. In the simplest case of a bit propagating in AdS_D, one finds the locally $Sp(2,R)$-invariant Lagrangian

$$S \;=\; \int DY^{iA}\, Y_{iA} \;, \tag{4}$$

where Y^{iA} $(i=1,2)$ are coordinates and momenta in the $(D+1)$-dimensional embedding space with signature $\eta_{AB} = (- - + ... +)$ and $DY^{iA} = dY^{iA} + \Lambda^{ij} Y_j^A$ denotes the $Sp(2,R)$-covariant derivative along the world line. The local $Sp(2,R)$ symmetry, with generators

$$K_{ij} \;=\; \frac{1}{2}\, Y_i^A\, Y_{jA} \;, \tag{5}$$

embodies the p-brane τ-diffeomorphisms and the constraints associated with the embedding of the conical limit of AdS_D in the target space. Multi-bit states with fixed numbers of bits are postulated to be exact states in the theory, in analogy with the matrix-model interpretation of the discretized membrane in flat space. A more detailed discussion of the open-string-like quantum mechanics of such systems will be presented in [15].

A crucial result is that the state space of a single bit,

$$\mathcal{H}_{\text{1-bit}} \;=\; \{\; |\Psi\rangle \;:\; K_{ij} \star |\Psi\rangle = 0 \;\} \;, \tag{6}$$

where $\star$ denotes the non-commutative and associative oscillator product, coincides with the CPT self-conjugate scalar *singleton* [18, 34, 15]

$$\mathcal{H}_{\text{1-bit}} = D(\epsilon_0;\{0\}) \oplus \widetilde{D}(\epsilon_0;\{0\}) \;, \qquad \epsilon_0 = \frac{1}{2}\,(D-3) \;, \tag{7}$$

where $D(E_0;S_0)$ and $\widetilde{D}(E_0;S_0)$ denote [3] lowest and highest weight spaces of the $SO(D-1,2)$ generated by

$$M_{AB} \;=\; \frac{1}{2}\,Y^i_A\,Y_{iB}\ . \tag{8}$$

The naively defined norm of the 1-bit states diverges, since the lowest/highest weight states $|\Omega_\pm\rangle$ are "squeezed" Y^i_A-oscillator excitations [34,15]. As we shall see, a related issue arises in the corresponding HS gauge theory, when it is subject to the strong $Sp(2,R)$-invariance condition (1).

The single-bit Hilbert space is irreducible under the combined action of $SO(D-1,2)$ and the discrete involution π defined by $\pi(|\Omega_\pm\rangle)=|\Omega_\mp\rangle$, and by

$$\pi(P_a) \;=\; -P_a\ , \qquad \pi(M_{ab}) \;=\; M_{ab}\ , \tag{9}$$

where $P_a=V_a{}^AV^BM_{AB}$ and $M_{ab}=V_a{}^AV_b{}^BM_{AB}$ are the (A)dS translation and rotation generators, and the tangent-space Lorentz index a is defined by $X_a=V_a{}^AX_A$ and $X=V^AX_A$, where X_A is any $SO(D-1,2)$ vector and $(V_a{}^A,V^A)$ is a quasi-orthogonal embedding matrix subject to the conditions

$$V_a{}^A\,V_b{}^B\,\eta_{AB} \;=\; \eta_{ab}\ , \qquad V_a{}^A\,V_A \;=\; 0\ , \qquad V^A\,V_A \;=\; -\,1\ . \tag{10}$$

The AdS generators, however, do not act transitively on the singleton weight spaces. The smallest Lie algebra with this property is the *minimal HS extension* $ho_0(D-1,2)$ *of* $SO(D-1,2)$ defined by [4] [35,36,11,34,15,17]

$$ho_0(D-1,2) \;=\; Env_1(SO(D-1,2))/\mathcal{I}\ , \tag{11}$$

where $Env_1(SO(D-1,2))$ is the subalgebra of $Env(SO(D-1,2))$ elements that are odd under the τ-map

$$\tau(M_{AB}) \;=\; -M_{AB}\ , \tag{12}$$

that may be regarded as an analog of the transposition of matrices and acts as an anti-involution on the enveloping algebra,

$$\tau(P\star Q) \;=\; \tau(Q)\star\tau(P)\ , \qquad P,Q\in Env(SO(D-1,2))\ , \tag{13}$$

and $\mathcal{I}$ is the subalgebra of $Env_1(SO(D-1,2))$ given by the annihilating ideal of the singleton

$$\mathcal{I} \;=\; \{P\in Env_1(SO(D-1,2))\ :\ P\star|\Psi\rangle=0 \text{ for all } |\Psi\rangle\in D(\epsilon_0;\{0\})\ \}\ . \tag{14}$$

The minimal HS algebra has the following key features [34,15]:

1. It admits a decomposition into *levels* labelled by irreducible finite-dimensional $SO(D-1,2)$ representations $\{2\ell+1,2\ell+1\}$ for $\ell=0,1,2,...$, with the 0-th level identified as the $SO(D-1,2)$ subalgebra.
2. It acts transitively on the scalar singleton weight spaces.
3. It is a minimal extension of $SO(D-1,2)$, in the sense that if $SO(D-1,2)\subseteq\mathcal{L}\subseteq ho_0(D-1,2)$ and $\mathcal{L}$ is a Lie algebra, then either $\mathcal{L}=SO(D-1,2)$ or $\mathcal{L}=ho_0(D-1,2)$.

One can show that these features determine uniquely the algebra, independently of the specific choice of realization [15]. In particular, in $D=4,5,7$ this implies the isomorphisms [34,15]

$$ho_0(3,2)\simeq hs(4)\ , \qquad ho_0(4,2)\simeq hs(2,2)\ , \qquad ho_0(6,2)\simeq hs(8^*)\ , \tag{15}$$

where the right-hand sides denote the spinor-oscillator realizations.

[3] We denote $SL(N)$ highest weights by the number of boxes in the rows of the corresponding Young tableaux, $(m_1,\dots,m_n)\equiv(m_1,\dots,m_n,0,\dots,0)$, and $SO(D-1,T)$ highest weights by $\{m_1,\dots,m_n\}\equiv\{m_1,\dots,m_n,0,\dots,0\}$, for $N=D+1$ or D, and $T=1$ or 2. The values of N and T will often be left implicit, but at times, for clarity, we shall indicate them by a subscript.

[4] This algebra is denoted by $hu(1/sp(2)[2,D-1])$ in [11].

¿From the space-time point of view, the dynamics of tensionless extended objects involves processes where multi-bit states interact by creation and annihilation of pairs of bits. Roughly speaking, unlike ordinary multi-particle states, multi-bit states have an extended nature that should reflect itself in a prescription for assigning weights $\hbar^L$ to the amplitudes in such a way that "hard" processes involving many simultaneous collisions be suppressed with respect to "soft" processes. This prescription leads to a (1+1)-dimensional topological Sp-gauged σ-model *à la* Cattaneo-Felder [39], whose associated Batalin-Vilkovisky master equation have a structure that might be related to the Vasiliev equations [15].

The 2-bit states are of particular interest, since it is natural to expect, in many ways, that their classical self-interactions form a *consistent truncation* of the classical limit of the full theory. The 2-bit Hilbert space consists of the symmetric (S) and anti-symmetric (A) products of two singletons, that decompose into *massless one-particle states* in AdS with even and odd spin [37, 34, 15]

$$D(\epsilon_0;\{0\})\otimes D(\epsilon_0;\{0\}) \;=\; \left[\bigoplus_{s=0,2,\dots} D(s+2\epsilon_0;\{s\})\right]_{\rm S}\oplus\left[\bigoplus_{s=1,3,\dots} D(s+2\epsilon_0;\{s\})\right]_{\rm A}\ . \tag{16}$$

The symmetric part actually coincides with the spectrum of the *minimal bosonic HS gauge theory* in D dimensions, to which we now turn our attention.

3 The off-shell theory

3.1 General Set-Up

The HS gauge theory based on the minimal bosonic HS algebra $ho_0(D-1,2)$, given in (11), is defined by a set of constraints on the curvatures of an *adjoint one form* $\widehat{A}$ and a *twisted-adjoint zero form* $\widehat{\Phi}$. The off-shell master fields are defined by expansions in the Y^i_A oscillators, obeying a *weak* $Sp(2,R)$ invariance condition that will be defined in eq. (33) below. The constraints reduce drastically the number of independent component fields without implying any on-shell field equations. The independent fields are a real scalar field, arising in the master 0-form, a metric vielbein and an infinite tower of symmetric rank-s tensors for $s=4,6,\dots$, arising in the master 1-form. The remaining components are auxiliary fields: in $\widehat{A}_\mu$ one finds the Lorentz connection and its HS counterparts, that at the linearized level reduce to the Freedman-de Wit connections in a suitable gauge to be discussed below; in $\widehat{\Phi}$ one finds the derivatives of the scalar field, the spin 2 Riemann tensor, its higher-spin generalizations, and all their derivatives. Therefore, the minimal theory is a HS generalization of Einstein gravity, with infinitely many bosonic fields but no fermions, and without an internal gauge group. We have already stressed that the minimal model admits generalizations with internal symmetry groups, that enter in way highly reminiscent of Chan-Paton groups [14] for open strings, but here we shall confine our attention to the minimal case.

The curvature constraints define a Cartan integrable system, a very interesting construction first introduced in supergravity, in its simplest non-conventional setting with 1-forms and 3-forms, by D'Auria and Fré [40]. Any such system is gauge invariant by virtue of its integrability, and is also manifestly diffeomorphism invariant since it is formulated entirely in terms of differential forms. The introduction of twisted-adjoint zero forms, however, was a key contribution of Vasiliev [16], that resulted in the emergence of the present-day *unfolded formulation*. The full HS gauge theory *a priori* does not refer to any particular space-time manifold, but the introduction of a D-dimensional bosonic spacetime $\mathcal{M}_D$ yields a weak-field expansion in terms of recognizable tensor equations. An illustration of the general nature of this setting is provided in [26], where 4D superspace formulations are constructed directly in unfolded form picking a superspace as the base manifold.

A key technical point of Vasiliev's construction is an internal noncommutative Z-space $\mathcal{M}_Z$, that, from the space-time viewpoint, may be regarded as a tool for obtaining a highly non-linear integrable system with 0-forms on a commutative space-time $\mathcal{M}_D$ [1] from a simple integrable system on a non-commutative extended space. Although apparently *ad hoc*, this procedure has a rather precise meaning within the BRST formulation of the phase-space covariant treatment of bits *à la* Kontsevich-Cattaneo-

Felder [39], as will be discussed in [15].

3.2 Oscillators and Master Fields

Following Vasiliev [11], we work with bosonic oscillators Y^i_A and Z^i_A, where $A = -1, 0, 1, \ldots, D-1$ labels an $SO(D-1,2)$ vector and $i = 1, 2$ labels an $Sp(2,R)$ doublet. The oscillators obey the associative $\star$-product algebra

$$Y^A_i \star Y^B_j \;=\; Y^A_i Y^B_j + i\epsilon_{ij}\eta^{AB} \;, \quad Y^A_i \star Z^B_j \;=\; Y^A_i Z^B_j - i\epsilon_{ij}\eta^{AB} \;, \tag{17}$$

$$Z^A_i \star Y^B_j \;=\; Z^A_i Y^B_j + i\epsilon_{ij}\eta^{AB} \;, \quad Z^A_i \star Z^B_j \;=\; Z^A_i Z^B_j - i\epsilon_{ij}\eta^{AB} \;, \tag{18}$$

where the products on the right-hand sides are Weyl ordered, *i.e.* symmetrized, and obey the conditions

$$(Y^i_A)^\dagger \;=\; Y^i_A \qquad \text{and} \qquad (Z^i_A)^\dagger \;=\; -Z^i_A \;. \tag{19}$$

It follows that the $\star$-product of two Weyl-ordered polynomials $\widehat{f}$ and $\widehat{g}$ can be defined by the integral

$$\widehat{f}(Y,Z) \star \widehat{g}(Y,Z) \;=\; \int \frac{d^{2(D+1)}S\, d^{2(D+1)}T}{(2\pi)^{2(D+1)}}\, \widehat{f}(Y+S, Z+S)\, \widehat{g}(Y+T, Z-T)\, e^{iT^{iA}S_{iA}} \;, \tag{20}$$

where S and T are real unbounded integration variables.

The master fields of the minimal model are a 1-form and a 0-form

$$\widehat{A} \;=\; dx^\mu \widehat{A}_\mu(x,Y,Z) + dZ^{iA}\widehat{A}_{iA}(x,Y,Z) \;, \qquad \widehat{\Phi} \;=\; \widehat{\Phi}(x,Y,Z) \;, \tag{21}$$

subject to the conditions defining the adjoint and twisted-adjoint representations,

$$\tau(\widehat{A}) \;=\; -\widehat{A} \;, \qquad \widehat{A}^\dagger \;=\; -\widehat{A} \;, \qquad \tau(\widehat{\Phi}) \;=\; \pi(\widehat{\Phi}) \;, \qquad \widehat{\Phi}^\dagger \;=\; \pi(\widehat{\Phi}) \;, \tag{22}$$

where π and τ, defined in (9) and (12), act on the oscillators as

$$\pi(\widehat{f}(x^\mu, Y^i_a, Y^i, Z^i_A, Z^i)) \;=\; \widehat{f}(x^\mu, Y^i_a, -Y^i, Z^i_a, -Z^i) \;, \tag{23}$$

$$\tau(\widehat{f}(x^\mu, Y^i_A, Z^i_A)) \;=\; \widehat{f}(x^\mu, iY^i_A, -iZ^i_A) \;. \tag{24}$$

The π-map can be generated by the $\star$-product with the hermitian and τ-invariant oscillator construct

$$\kappa \;=\; e^{iZ^iY_i} \;=\; (\kappa)^\dagger \;=\; \tau(\kappa) \;, \tag{25}$$

such that

$$\kappa \star \widehat{f}(Y^i, Z^i) \;=\; \pi(\widehat{f}(Y^i, Z^i)) \star \kappa \;=\; \kappa \widehat{f}(Z^i, Y^i) \;. \tag{26}$$

Strictly speaking, κ lies outside the domain of "arbitrary polynomials", for which the integral representation (20) of the $\star$-product is obviously well-defined. There is, however, no ambiguity in (26), in the sense that expanding κ in a power series and applying (20) term-wise, or making use of the standard representation of the Dirac δ function, one is led to the same result. This implies, in particular, that $\kappa \star \kappa = 1$, so that $\kappa \star \widehat{f} \star \kappa = \pi(\widehat{f})$, although the form (26) will be most useful in the formal treatment of the master constraints. It is also worth stressing that the n-th order curvature corrections, to be discussed later, contain n insertions of exponentials, of the type [1]

$$\cdots \star \kappa(t_1) \star \ldots \star \kappa(t_n) \cdots \;, \qquad \kappa(t) \;=\; e^{itZ^iY_i} \;, \quad t_i \in [0,1] \;, \tag{27}$$

that are thus well-defined[5], and can be expanded term-wise, as was done for instance to investigate the

[5]We are grateful to M. Vasiliev for an extensive discussion on these points during the Brussels Workshop.

second-order scalar corrections to the stress energy tensor in [41], or the scalar self-couplings in [42].

3.3 Master Constraints

The off-shell minimal bosonic HS gauge theory is defined by

i) the integrable curvature constraints

$$\widehat{F} \;=\; \frac{i}{2}\, dZ^i \wedge dZ_i\, \widehat{\Phi} \star \kappa\;, \qquad \widehat{D}\widehat{\Phi} \;=\; 0\;, \tag{28}$$

where $Z^i = V^A Z^i_A$, the curvature and the covariant derivative are given by

$$\widehat{F} \;=\; d\widehat{A} + \widehat{A} \star \wedge \widehat{A}\;, \qquad \widehat{D}\widehat{\Phi} \;=\; d\widehat{\Phi} + [\widehat{A}, \widehat{\Phi}]_\pi\;, \tag{29}$$

and the π-twisted commutator is defined as

$$[\widehat{f}, \widehat{g}]_\pi = \widehat{f} \star \widehat{g} - \widehat{g} \star \pi(\widehat{f})\;. \tag{30}$$

The integrability of the constraints implies their invariance under the general gauge transformations

$$\delta_{\widehat{\epsilon}} \widehat{A} \;=\; \widehat{D}\widehat{\epsilon}\;, \qquad \delta_{\widehat{\epsilon}} \widehat{\Phi} \;=\; -[\widehat{\epsilon}, \widehat{\Phi}]_\pi\;, \tag{31}$$

where the covariant derivative of an adjoint element is defined by $D\widehat{\epsilon} = d\widehat{\epsilon} + \widehat{A} \star \widehat{\epsilon} - \widehat{\epsilon} \star \widehat{A}$;

ii) the invariance of the master fields under global $Sp(2, R)$ gauge transformations with $\widehat{\epsilon}(\lambda) = \frac{i}{2}\lambda^{ij}\widehat{K}_{ij}$, where the λ^{ij} are constant parameters, and [11]

$$\widehat{K}_{ij} \;=\; K_{ij} + \frac{1}{2}\left(\widehat{S}^A_{(i} \star \widehat{S}_{j)A} - Z^A_i Z_{jA}\right)\;, \quad \widehat{S}^A_i \equiv Z^A_i - 2i\widehat{A}^A_i\;, \tag{32}$$

where K_{ij} are the $Sp(2, R)$ generators of the linearized theory, defined in (5).

The $Sp(2, R)$ invariance conditions can equivalently be written in the form

$$[\widehat{K}_{ij}, \widehat{\Phi}]_\pi \;=\; 0\;, \qquad \widehat{D}\widehat{K}_{ij} \;=\; 0\;. \tag{33}$$

These conditions remove all component fields that are not singlets under $Sp(2, R)$ transformations. We also stress that the $\widehat{S} \star \widehat{S}$ terms play a crucial role in the $Sp(2, R)$ generators $\widehat{K}_{ij}$ of (32): without them all $Sp(2, R)$ indices originating from the oscillator expansion would transform canonically but, as shown in [11], they guarantee that the same holds for the doublet index of $\widehat{A}_{A\,i}$.

3.4 Off-Shell Adjoint and Twisted-Adjoint Representations

The gauge transformations (31) are based on a rigid Lie algebra that we shall denote by $ho(D-1, 2)$, and that can be defined considering x- and Z-independent gauge parameters. Consequently, this algebra is defined by [11]

$$ho(D-1, 2) \;=\; \Big\{Q(Y):\ \tau(Q) = Q^\dagger = -Q\;, \quad [K_{ij}, Q]_\star = 0\Big\}\;, \tag{34}$$

with Lie bracket $[Q, Q']_\star$, and where the linearized $Sp(2, R)$ generators are defined in (5). The τ-condition implies that an element Q admits the level decomposition $Q = \sum_{\ell=0}^\infty Q_\ell$, where $Q_\ell(\lambda Y) = \lambda^{4\ell+2} Q_\ell(Y)$ and $[K_{ij}, Q_\ell]_\star = 0$. As shown in [11], the $Sp(2, R)$-invariance condition implies that $Q_\ell(Y)$ has a Y-expansion in terms of (traceless) $SO(D-1, 2)$ tensors combining into single $SL(D+1)$ tensors with

highest weights corresponding to Young tableaux of type $(2\ell+1, 2\ell+1)$:

$$Q_\ell \;=\; \frac{1}{2i}\, Q^{(2\ell+1,2\ell+1)}_{A_1\dots A_{2\ell+1},B_1\dots B_{2\ell+1}} M^{A_1B_1}\cdots M^{A_{2\ell+1}B_{2\ell+1}}\omega 2 \qquad (35)$$
$$\;=\; \frac{1}{2i}\sum_{m=0}^{2\ell+1} Q^{(2\ell+1,m)}_{a_1\dots a_{2\ell+1}b_1\cdots b_m} M^{a_1b_1}\cdots M^{a_mb_m}P^{a_{m+1}}\cdots P^{a_{2\ell+1}}\ ,$$

where the products are Weyl ordered and the tensors in the second expression are labelled by highest weights of $SL(D)$. To reiterate, the algebra $ho(D-1,2)$ is a reducible HS extension of $SO(D-1,2)$, whose ℓ-th level generators fill a *finite-dimensional* reducible representation of $SO(D-1,2)$. In the next section we shall discuss how to truncate the trace parts at each level in $ho(D-1,2)$ to define a subalgebra of direct relevance for the on-shell theory.

The 0-form master field Φ, defined in (38) below, belongs to the twisted-adjoint representation $T[ho(D-1,2)]$ of $ho(D-1,2)$ associated with the rigid covariantization terms in $\widehat{D}_\mu\widehat{\Phi}|_{Z=0}$. Consequently,

$$T[ho(D-1,2)] \;=\; \Big\{R(Y):\ \tau(R)=R^\dagger=\pi(R)\ ,\quad [K_{ij},R]_\star=0\Big\}\ , \qquad (36)$$

on which $ho(D-1,2)$ acts via the π-twisted commutator $\delta_\epsilon R=-[\epsilon,R]_\pi$. A twisted-adjoint element admits the level decomposition $R=\sum_{\ell=-1}^\infty R_\ell$, where R_ℓ has an expansion in terms of $SL(D)$ irreps as $(\ell\geq -1)$

$$R_\ell \;=\; \sum_{k=0}^\infty \frac{1}{k!}\, R^{(2\ell+2+k,2\ell+2)}_{a_1\dots a_{2\ell+2+k},b_1\dots b_{2\ell+2}} M^{a_1b_1}\cdots M^{a_{2\ell+2}b_{2\ell+2}}P^{a_{2\ell+3}}\cdots P^{a_{2\ell+2+k}}\ . \qquad (37)$$

It can be shown that *each level forms a separate, infinite dimensional reducible $SO(D-1,2)$ representation*, that includes an infinity of trace parts that will be eliminated in the on-shell formulation. In particular, R_{-1} has an expansion in terms of $SL(D)$ tensors carrying the same highest weights as an off-shell scalar field and its derivatives, while each of the R_ℓ, $\ell\geq 0$, consists of $SL(D)$ tensors corresponding to an off-shell spin-$(2\ell+2)$ Riemann tensor and its derivatives.

3.5 Weak-Field Expansion and Linearized Field Equations

The master constraints (28) can be analyzed by a weak-field expansion, which will sometimes be referred to as a *perturbative expansion*, in which the scalar field, the HS gauge fields and all curvatures (including the spin-two curvature) are indeed treated as weak. One can then start from the initial conditions

$$A_\mu \;=\; \widehat{A}_\mu|_{Z=0}\ ,\qquad \Phi \;=\; \widehat{\Phi}|_{Z=0}\ , \qquad (38)$$

fix a suitable gauge and solve for the Z-dependence of $\widehat{A}$ and $\widehat{\Phi}$ order by order in Φ integrating the constraints

$$\widehat{D}_i\widehat{\Phi} \;=\; 0\ ,\qquad \widehat{F}_{ij} \;=\; -i\epsilon_{ij}\widehat{\Phi}\star\kappa\ ,\qquad \widehat{F}_{i\mu} \;=\; 0\ , \qquad (39)$$

thus obtaining, schematically,

$$\widehat{\Phi} \;=\; \widehat{\Phi}(\Phi)\ ,\qquad \widehat{A}_\mu \;=\; \widehat{A}_\mu(\Phi,A_\mu)\ ,\qquad \widehat{A}_i \;=\; \widehat{A}_i(\Phi)\ . \qquad (40)$$

Substituting these solutions in the remaining constraints evaluated at $Z=0$ gives

$$\widehat{F}_{\mu\nu}|_{Z=0} \;=\; 0\ ,\qquad \widehat{D}_\mu\widehat{\Phi}|_{Z=0} \;=\; 0\ , \qquad (41)$$

that describe the full non-linear off-shell HS gauge theory in ordinary spacetime, *i.e. the full set of non-linear constraints that define its curvatures without implying any dynamical equations.*

A few subtleties are involved in establishing the integrability in spacetime of (41) in perturbation theory. One begins as usual by observing that integrability holds to lowest order in Φ, and proceeds by assuming that *all* the constraints hold for *all* x^μ and Z to n-th order in Φ. One can then show that the

constraint $(\widehat{D}_i\widehat{\Phi})^{(n+1)}=0$ is an integrable partial differential equation in Z for $\widehat{\Phi}^{(n+1)}$, whose solution obeys $\partial_i(\widehat{D}_\mu\widehat{\Phi})^{(n+1)}=0$, which in its turn implies that $(\widehat{D}_\mu\widehat{\Phi})^{(n+1)}=0$ for all Z if $(\widehat{D}_\mu\widehat{\Phi})^{(n+1)}|_{Z=0}=0$. Proceeding in this fashion, one can obtain $\widehat{A}_i^{(n+1)}$, and then $\widehat{A}_\mu^{(n+1)}$, via integration in Z, to show that $\partial_i\widehat{F}_{\mu\nu}^{(n+1)}=0$, so that if $\widehat{F}_{\mu\nu}^{(n+1)}|_{Z=0}=0$, then $\widehat{F}_{\mu\nu}^{(n+1)}=0$ for all Z. It thus follows, by induction, that once the constraints (41) on $\widehat{F}_{\mu\nu}$ and $\widehat{\Phi}$ are imposed at $Z=0$, they hold for all Z, and are manifestly integrable on spacetime, since Z can be treated as a parameter. Hence, their restriction to $Z=0$ is also manifestly integrable in spacetime, simply because the exterior derivative $dx^\mu\partial_\mu$ does not affect the restriction to $Z=0$.

Having obtained (41) in a Φ-expansion, one can write

$$A_\mu = e_\mu+\omega_\mu+W_\mu \ , \tag{42}$$

where $e_\mu=\frac{1}{2i}\,e_\mu{}^aP_a$ and $\omega_\mu=\frac{1}{2i}\,\omega_\mu{}^{ab}M_{ab}$, and where W_μ contains the HS gauge fields residing at levels $\ell\geq 1$. We would like to stress that until now μ has been treated as a formal curved index with no definite intrinsic properties. Treating e_μ and ω_μ as strong fields and referring μ explicitly to a D-dimensional bosonic spacetime clearly builds a perturbative expansion that preserves local Lorentz invariance and D-dimensional diffeomorphism invariance. It would be interesting to investigate to what extent the higher-spin geometry could be made more manifest going beyond this choice.

To first order in the weak fields Φ and W_μ, the constraints (41) reduce to

$$\mathcal{R}+\mathcal{F} \;=\; i\, e^a\wedge e^b\,\left.\frac{\partial^2\Phi}{\partial Y^{ai}\partial Y^b_i}\right|_{Y^i=0} \ , \tag{43}$$

$$\nabla\Phi+\frac{1}{2i}\,e^a\{P_a,\Phi\}_\star \;=\; 0 \ , \tag{44}$$

where $\mathcal{R}$ is the $SO(D-1,2)$-valued curvature of $E\equiv e+\omega$ defined by $\mathcal{R}=dE+[E,E]_\star$, and $\mathcal{F}$ is the linearized $SO(D-1,2)$ covariant curvature defined by $\mathcal{F}=dW+\{E,W\}_\star$, and $\nabla\Phi=d\Phi+[\omega,\Phi]_\pi$ is the Lorentz covariant derivative of Φ. Since each level of the adjoint and twisted-adjoint master fields forms a separate representation of $SO(D-1,2)$, the linearized constraints split into independent sets for the individual levels:

$$\ell=0 \quad : \quad \mathcal{R} \;=\; -8\,i\,e^a\wedge e^b\;\Phi^{(2,2)}_{ac,bd}\,M^{cd} \ , \tag{45}$$

$$\ell\geq 1 \quad : \quad \mathcal{F}_\ell \;=\; i\,\epsilon^a\wedge e^b\;\frac{\partial^2\Phi_\ell^{(2\ell+2,2\ell+2)}}{\partial Y^{ai}\partial Y^b_i} \ , \tag{46}$$

$$\ell\geq -1 \quad : \quad \nabla\Phi_\ell+\frac{1}{2i}\,e^a\,\{P_a,\Phi_\ell\}_\star \;=\; 0 \ . \tag{47}$$

Note that $\mathcal{F}$ is to be expanded as in (35), and Φ_ℓ as in (37). Furthermore, in order to compute the star anticommutator in the last equation, one must use the Y-expansion of all generators involved, and double contractions contribute to this term. Note also, for example, that even if the star commutator occurs in $\mathcal{R}$, using the commutation relation between the AdS generators is not sufficient, and one must also recall relations such as $[M_{[ab},P_{c]}]=0$, that follow from the oscillator realization of these generators.

The first of (47) contains the usual torsion constraint, and identifies $\Phi^{(2,2)}_{ab,cd}$ with the $SO(D-1,2)$-covariantized Riemann curvature of $e_\mu{}^a$. As $\Phi^{(2,2)}_{ab,cd}$ is traceful, these equations describe off-shell AdS gravity: the trace of (45) simply determines the trace part of $\Phi^{(2,2)}_{ab,cd}$, rather than giving rise to the Einstein equation. This generalizes to the higher levels, and a detailed analysis of (46) reveals that [17]:

i) the gauge parameter ϵ_ℓ contains Stückelberg-type shift symmetries;

ii) W_ℓ contains pure gauge parts and auxiliary gauge fields that can be eliminated using shift symmetries or constraints on torsion-like components of $\mathcal{F}_\ell$, respectively;

iii) the remaining independent components of W_ℓ correspond to the fully symmetric tensors

$$\phi^{(s)}_{a_1\dots a_s} \equiv e_{(a_1}{}^{\mu}\, W^{(s-1)}_{\mu,a_2\dots a_s)} \ , \qquad s = 2\ell+2 \ ; \tag{48}$$

iv) the system is off-shell: the remaining non-torsion-like components of $\mathcal{F}_\ell$ *vanish identically*, with the only exception of the s-th one, that defines the generalized (traceful) Riemann tensor of spin $s = 2\ell+2$,

$$R^{(s,s)}_{a_1\dots a_s,}{}^{b_1\dots b_s} \equiv e^{\mu}_{(a_1} e^{\nu(b_1} \mathcal{F}^{(s-1,s-1)}_{\mu\nu,a_2\dots a_s),}{}^{b_2\dots b_s)} = 4s^2 \Phi^{(s,s)}_{a_1\dots a_s,}{}^{b_1\dots b_s} \ , \tag{49}$$

where the identification follows from (43) and the Riemann tensor is built from s derivatives of $\phi^{(s)}$.

Turning to the Φ_ℓ-constraint (47), one can show that its component form reads ($s = 2\ell+2$, $\ell \geq -1$, $k \geq 0$):

$$\nabla_\mu \Phi^{(s+k,s)}_{a_1\dots a_{s+k},b_1\dots b_s} = \frac{i}{4}\,(s+k+2)\Phi^{(s+k+1,s)}_{\underbrace{\mu a_1\dots a_{s+k},b_1\dots b_s}_{(s+k,s)}} + i\,\frac{(k+1)(s+k)}{s+k+1}\ \eta_{\mu(a_1}\Phi^{(s+k-1,s)}_{a_2\dots a_{s+k}),b_1\dots b_s} \ , \tag{50}$$

where we have indicated a Young projection to the tableaux with highest weight $(s+k,s)$. Symmetrizing μ and $a_1\dots a_k$ shows that $\Phi^{(s+k+1,s)}$ ($k \geq 0$) are auxiliary fields, expressible in terms of derivatives of $\Phi^{(s,s)}$. The $\Phi^{(s,s)}$ components are generalized (traceful) Riemann tensors given by (49) for $\ell \geq 0$, and an independent scalar field for $\ell = -1$,

$$\phi \equiv \Phi^{(0,0)} \ . \tag{51}$$

The remaining components of (50), given by the $(s+k,s+1)$ and $(s+k,s,1)$ projections, are Bianchi identities. Hence, no on-shell conditions are hidden in (50).

In particular, combining the $k = 0,1$ components of (50) for $s = 0$, one finds

$$\left(\nabla^2 + \frac{D}{2}\right)\phi = -\frac{3}{8}\,\eta^{ab}\,\Phi^{(2,0)}_{ab} \ , \tag{52}$$

which, as stated above, determines the trace part $\eta^{ab}\Phi^{(2)}_{ab}$ of the auxiliary field $\Phi^{(2)}_{ab}$ rather than putting the scalar field ϕ on-shell. Similarly, the $s \geq 2$ and $k = 0,1$ components of (50) yield

$$\nabla^2\Phi^{(s,s)}_{a_1\dots a_s,b_1\dots b_s} + \frac{(D+s)}{2}\,\Phi^{(s,s)}_{a_1\dots a_s,b_1\dots b_s} = \frac{(s+2)(s+3)}{16}\,\eta^{\mu\nu}\Phi^{(s+2,s)}_{\mu\nu\underbrace{a_1\dots a_s,b_1\dots b_s}_{(s,s)}} \ , \tag{53}$$

that determine the trace parts of the auxiliary fields $\Phi^{(s+2,s)}$, rather than leading to the usual Klein-Gordon-like equations satisfied by on-shell Weyl tensors.

In summary, the constraints (28) describe an off-shell HS multiplet with independent field content given by a tower of real and symmetric rank-s $SL(D)$ tensors with $s = 0,2,4,\dots$,

$$\phi \ , \ e_\mu{}^a \ , \ \phi^{(s)}_{a_1\dots a_s} \quad (s = 4,6,\dots) \ , \tag{54}$$

where the scalar field ϕ is given in (51), the vielbein $e_\mu{}^a$ is defined in (42), and the real symmetric HS tensor fields are given in (48).

4 The on-shell theory

4.1 On-Shell Projection

It should be appreciated that, if the trace parts in (52) and (53) were simply dropped, *the resulting masses would not coincide with the proper values for a conformally coupled scalar and on-shell spin-s Weyl tensors in D dimensions.* Hence, the trace parts contained in Φ must be carefully eliminated, and cannot be simply set equal to zero. To this end, as discussed in the Introduction, motivated by the arguments based on $Sp(2,R)$-gauged noncommutative phase spaces arising in the context of tensionless strings and two-time physics, we would like to propose the on-shell projection

$$\widehat{K}_{ij} \star \widehat{\Phi} \ = \ 0 \ , \tag{55}$$

where $\widehat{K}_{ij}$ is defined in (32), be adjoined to the master constraints of [11]

$$\widehat{F} \ = \ \frac{i}{2}\, dZ^i \wedge dZ_i \ \widehat{\Phi} \star \kappa \ , \qquad \widehat{D}\widehat{\Phi} \ = \ 0 \ . \tag{56}$$

Although one can verify that the combined constraints (55) and (56) remain formally integrable at the full non-linear level, the strong $Sp(2,R)$ projection (55) should be treated with great care, since *its perturbative solution involves non-polynomial functions of the oscillators associated with projectors whose products can introduce divergences in higher orders of the perturbative expansion unless they are properly treated* [34]. It is important to stress, however, that these singularities draw their origin from the curvature expansion, and in Section 5 we shall describe how a finite curvature expansion might be defined.

We can thus begin by exploring the effects of (55) on the linearized field equations and, as we shall see, the correct masses emerge in a fashion which is highly reminiscent of what happens in spinor formulations. There is a further subtlety, however. According to the analysis in the previous section, if the internal indices of the gauge fields in $\widehat{A}$ were also taken to be traceless, the linearized 2-form constraint (46) would reduce to the Fronsdal equations [23], that in the index-free notation of [21] read

$$\mathcal{F} \ = \ 0 \ , \tag{57}$$

where the Fronsdal operator is

$$\mathcal{F} \ \equiv \ \nabla^2\, \phi \ - \ \nabla\,\nabla\cdot\phi \ + \ \nabla\,\nabla\,\phi^{'} \ - \ \frac{1}{L^2}\,\{[(3-D-s)(2-s)-s]\phi + 2g\phi^{'}\} \ , \tag{58}$$

and where "primes" denote traces taken using the AdS metric g of radius L. We would like to stress that this formulation is based on *doubly traceless* gauge fields, and is invariant under gauge transformations with *traceless* parameters. However, the constrained gauge fields of this conventional formulation should be contrasted with those present in (56), namely the metric and the HS gauge fields collected in (48). These fields do contain trace parts, and enforcing the 0-form projection as in (55), as we shall see, actually leaves the gauge fields free to adjust themselves to their constrained sources in the projected Weyl 0-form $\widehat{\Phi}$, thus extending the conventional Fronsdal formulation based on (57) and (58) to the geometric formulation of [21]. This is due to the fact that, once the 0-forms on the right-hand side of (46) are constrained to be traceless Weyl tensors, certain trace parts of the spin-s gauge fields can be expressed in terms of gradients of traceful rank-$(s-3)$ symmetric tensors $\alpha_{a_1\ldots a_{s-3}}$. As we shall see in detail in subsection 4.4, up to the overall normalization of α, that is not chosen in a consistent fashion throughout this paper, these will enter the physical spin-s field equations embodied in the Vasiliev constraints precisely as in [21, 22]. In the index-free notation of [21], the complete linearized field equations in an AdS background would thus read

$$\mathcal{F} \ = \ 3\,\nabla\nabla\nabla\alpha \ - \ \frac{4}{L^2}\, g\,\nabla\alpha \ , \tag{59}$$

where $\mathcal{F}$ denotes again the Fronsdal operator. Notice, however, that *now the gauge field ϕ is traceful and α plays the role of a compensator for the traceful gauge transformations*,

$$\begin{aligned} \delta\,\phi &= \nabla\Lambda\ , \\ \delta\,\alpha &= \Lambda^{'}\ , \end{aligned} \tag{60}$$

with Λ' the trace of the gauge parameter Λ. When combined with the Bianchi identity, eq. (59) implies that [22]

$$\phi'' \;=\; 4\nabla\cdot\alpha + \nabla\alpha'\ , \tag{61}$$

so that once α is gauged away using Λ', eq. (59) reduces to the Fronsdal form. It will be interesting to explore the role of the additional gauge symmetry in the interactions of the model.

To reiterate, the system (55) and (56) provides a realization *à la* Cartan of an on-shell HS gauge theory that embodies the non-local geometric equations of [21], with traceful gauge fields and parameters, rather than the more conventional Fronsdal form [23].

4.2 On-Shell Adjoint and Twisted-Adjoint Representations

It is important to stress that in our proposal both the on-shell and off-shell systems contain a gauge field in the adjoint representation of $ho(D-1,2)$, while in the on-shell system the 0-form obeys an additional constraint, the strong $Sp(2,R)$ projection condition (55). The *on-shell twisted-adjoint representation* is thus defined for $D\geq 4$ by[6]

$$T_0[ho(D-1,2)] \;=\; \{S\in T[ho(D-1,2)]\ :\ K_{ij}\star S=0\ ,\quad S\star K_{ij}=0\}\ , \tag{62}$$

where the linearized $Sp(2,R)$ generators K_{ij} are defined in (5). The part of A_μ that annihilates the twisted-adjoint representation can thus be removed from the rigid covariantizations in $\widehat{D}_\mu\widehat{\Phi}|_{Z=0}$. It forms an $Sp(2,R)$-invariant ideal $I(K)\subset ho(D-1,2)$ consisting of the elements generated by left or right $\star$-multiplication by K_{ij}, *i.e.*

$$I(K) \;=\; \Big\{K^{ij}\star f_{ij} : \tau(f_{ij})=(f_{ij})^\dagger=-f_{ij},\ [K_{ij},f^{kl}]_\star=4i\delta_{(i}^{(k}f_{j)}^{l)}\Big\}\ . \tag{63}$$

Factoring out the ideal $I(K)$ from the Lie bracket $[Q,Q']_\star$, one is led to the *minimal bosonic HS algebra*

$$ho(D-1,2)/I(K) \;\simeq\; ho_0(D-1,2) \;\equiv\; Env_1(SO(D-1,2))/\mathcal{I}\ , \tag{64}$$

defined in (11). The isomorphism follows from the uniqueness of the minimal algebra [15], and from the fact that $ho(D-1,2)/I(K)$ shares its key properties [11, 34, 15]. Thus, at the linearized level, the gauging of $ho_0(D-1,2)$ gives rise, for each even spin s, to the canonical frame fields [38]

$$A^{\{s-1,s-1\}}_{\mu,A_1\ldots A_{s-1},B_1\ldots B_{s-1}} \;=\; \Big\{A^{\{s-1,k\}}_{\mu,a_1\ldots a_{s-1},b_1\ldots b_k}\Big\}_{k=0}^{s-1}\ , \tag{65}$$

required to describe massless spin-s degrees of freedom in the conventional Fronsdal form.

We would like to stress, however, that at the non-linear level the *full* on-shell constraints (55) and (56) make use of the larger, reducible set of off-shell gauge fields valued in $ho(D-1,2)=ho_0(D-1,2)\oplus I(K)$. Hence, while the linearized compensator form (59) can be simply gauge fixed to the conventional Fronsdal form, interesting subtleties might well arise at the nonlinear level. It would thus be interesting to compare the interactions defined by (55) and (56), to be extracted using the prescription of Section 5.3, with those resulting from the formulation in [11].

The minimal algebra allows matrix extensions that come in three varieties, corresponding to the three infinite families of classical Lie algebras [34]. We have already stressed that these enter in a fashion highly reminiscent of how Chan-Paton factors enter open strings [14], and in this respect the presently known

[6] In $D=3$ one can show that S is actually a constant [34, 17].

HS gauge theories appear more directly related to open than to closed strings. The minimal bosonic HS gauge theory has in fact a clear open-string analog, the $O(1)$ bosonic model described in the review paper of Schwarz in [14]. More support for this view will be presented in [15].

Dressing Functions

The strong $Sp(2,R)$-invariance condition on the twisted-adjoint representation $T_0[ho(D-1,2)]$, $K_{ij}\star S = 0$, or equivalently $K_I \star S = 0$, where I is a triplet index, can be formally solved letting [34]

$$S \;=\; M \star R \;, \tag{66}$$

where $R \in T[ho(D-1,2)]/I(K)$ and M is a function of $K^2 = K^I K_I$ that is *analytic* at the origin and satisfies

$$K_{ij} \star M \;=\; 0 \;, \qquad \tau(M) \;=\; M^\dagger \;=\; M \;. \tag{67}$$

This and the normalization $M(0) = 1$ imply that

$$M(K^2) \;=\; \sum_{p=0}^{\infty} \frac{(-4K^2)^p}{p!} \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{1}{2}(D+2p))} \;=\; \Gamma\left(\frac{D}{2}\right) \frac{J_{\frac{D}{2}-1}(4\sqrt{K^2})}{(2\sqrt{K^2})^{\frac{D}{2}-1}} \;, \tag{68}$$

where J is a Bessel function and Γ is the Euler Γ function. Actually, M belongs to a class of *dressing functions*

$$F(N;K^2) \;=\; \sum_{p=0}^{\infty} \frac{(-4K^2)^p}{p!} \frac{\Gamma(\frac{1}{2}(N+D))}{\Gamma(\frac{1}{2}(N+D+2p))} \;=\; \Gamma(\nu+1) \frac{J_\nu(4\sqrt{K^2})}{(2\sqrt{K^2})^\nu} \;, \tag{69}$$

related to Bessel functions of order $\nu = \frac{N+D-2}{2}$ and argument $4\sqrt{K^2}$, and in particular

$$M(K^2) \;=\; F(0;K^2) \;. \tag{70}$$

The dressing functions arise in the explicit level decomposition of the twisted-adjoint element S in (66). In fact, for $D \geq 4$ one finds

$$S = \sum_{\ell=-1}^{\infty} S_\ell \;, \qquad S_\ell = \sum_{q=0}^{\ell+1} S_{\ell,q} \;, \tag{71}$$

where the expansion of $S_{\ell,q}$ is given by ($s = 2\ell+2$) [17]

$$\begin{aligned} S_{\ell,q} \;=\; & \sum_{k=0}^{\infty} \frac{d_{s,k,q}}{k!} S^{\{s+k,s-2q\}}_{a_1\dots a_{s+k},b_{2q+1}\dots b_s} \eta_{b_1 b_2} \cdots \eta_{b_{2q-1}b_{2q}} F(2(s+k);K^2) \times \\ & \times M^{a_1 b_1} \cdots M^{a_s b_s} P^{a_{s+1}} \cdots P^{a_{s+k}} \;. \end{aligned} \tag{72}$$

The coefficients $d_{s,k,q}$ with $q \geq 1$ are fixed by the requirement that all Lorentz tensors arise from the decomposition of AdS tensors, $S^{\{s+k,s\}} \in S^{\{s+k,s+k\}_{D+1}}$, while $d_{s,k,0}$ can be set equal to one by a choice of normalization [17]. The Lorentz tensors arising in $S_{\ell,q}$ with $q \geq 1$ are simply combinations[7] of those in $S_{\ell,0}$, that constitute the various levels of the on-shell twisted-adjoint representation ($s = 2\ell+2$):

$$S_{\ell,0} \;:\; \left\{ S^{\{s+k,s\}}_{a_1\dots a_{s+k},b_1\dots b_s} \; (k = 0,1,2,\dots) \right\} \;. \tag{73}$$

The ℓ-th level forms an irreducible multiplet within the twisted-adjoint representation of $SO(D-1,2)$. In showing this explicitly, the key point is that the translations, generated by $\delta_\xi = \xi^a \{P_a, \cdots\}_\star$, do not mix different levels [17]. Thus, $S_{-1,0}$ affords an expansion in terms of Lorentz tensors carrying the

[7] As we shall see in the next subsection, this mixing has a sizable effect on the free field equations.

same highest weights as an on-shell scalar field and its derivatives, while the $S_{\ell,0}$, $\ell \geq 0$, correspond to on-shell spin-$(2\ell+2)$ Weyl tensors and their derivatives.

4.3 Linearized Field Equations and Spectrum of the Model

In order to obtain the linearized field equations, it suffices to consider the initial conditions at $Z = 0$ to lowest order in the 0-form[8]

$$\widehat{A}_\mu\Big|_{Z=0} = A_\mu \in ho(D-1,2) \ , \quad \widehat{\Phi}\Big|_{Z=0} = M \star C + \mathcal{O}(C^2) \ , \tag{74}$$

where $M \star C \in T_0[ho(D-1,2)]$. The first C correction to $\widehat{A}_\mu$ is then obtained integrating in Z the constraint $\widehat{F}_{i\mu} = 0$. Expanding also in the higher-spin gauge fields as discussed below (42) and fixing suitable gauges [11, 17], one finds that the Vasiliev equations reduce to [17]

$$\mathcal{R} + \mathcal{F} = i \left. \frac{\partial^2 (C \star M)}{\partial Y^{\mu i} \partial Y_i^\nu} \right|_{Y^i=0} , \omega 4 \tag{75}$$

$$\nabla(C \star M) + \frac{1}{2i} e^a \{P_a, C \star M\}_\star = 0 \ . \tag{76}$$

Φ Constraint and Role of the Dressing Functions

The expansion of the master field $C \star M$ involves the dressing functions $F(N; K^2)$, as in (72). These, in their turn, play a crucial role in obtaining the appropriate field equations already at the linearized level, since they determine the trace parts of the auxiliary fields $\Phi^{(s+2,s)}$ in (53). Using these traces in the iterated form of (76), one can show that the field equations for the scalar $\phi = C|_{Y=0}$ and the Weyl tensors $C^{\{s,s\}}$ are finally

$$(\nabla^2 - m_0^2)\,\phi = 0 \ , \qquad (\nabla^2 - m_s^2)\, C^{\{s,s\}}_{a_1 \ldots a_s, b_1 \ldots b_s} = 0 \ , \qquad m_s^2 = -\frac{1}{2}\,(s + D - 3) \ , \tag{77}$$

and contain the proper mass terms. The details of the computations leading to this result will be given in [17], but the effect on the scalar equation is simple to see and can be spelled out in detail here. Indeed, the relevant contributions to $K_{ij} \star \Phi$ can be traced to the action of K_{ij} on $(\phi + \frac{1}{2}\,\Phi^{(2,0)}_{ab} P^a P^b)$, that leads, after expanding the $\star$-products, to a term proportional to $K_{ij}\left(\phi + \frac{1}{4}\,\eta^{ab}\,\Phi^{(2,0)}_{ab}\right)$. This disappears precisely if $\eta^{ab}\,\Phi^{(2,0)}_{ab} = -4\,\phi$, which leads, via eq. (52), to the correct scalar mass term.

Once the mass terms are obtained, it is straightforward to characterize the group theoretical content of the spectrum, as we shall do in subsection 4.3.3.

$\mathcal{F}$ Constraint and Mixing Phenomenon

Since on shell and after gauge fixing $\mathcal{F} \in ho(D-1,2)/I(K)$, its expansion takes the form (35), in which the (traceful) $SL(D)$ representations $Q^{(2\ell+1,m)}$ must be replaced by the (traceless) $SO(D-1,1)$ representations $Q^{\{2\ell+1,m\}}$. Using this expansion for $\mathcal{F}$, and the expansions (71) and (72) for $C \star M$, an analysis of the $\mathcal{F}$ constraint (75) shows that the physical fields at level ℓ present themselves in an admixture with lower levels. Still, the integrability of the constraints ensures that diagonalization is possible, and we have verified this explicitly for the spin-2 field equation. Indeed, starting from (75) one finds

$$R_{\mu\nu} + \tfrac{1}{4}(D-1)g_{\mu\nu} = -\tfrac{16}{3}\nabla_{\{\mu}\nabla_{\nu\}}\phi + \tfrac{8(D-1)}{D}\, g_{\mu\nu}\,\phi \ , \tag{78}$$

where $R_{\mu\nu}$ denotes the spin-2 Ricci tensor for the metric $g_{\mu\nu} = e_\mu{}^a e_{\nu\, a}$ and ϕ is the physical scalar. Using the scalar field equation (77), one can show that the rescaled metric

$$\tilde{g}_{\mu\nu} = e^{-2u}\, g_{\mu\nu} \ , \qquad \text{with} \quad u = \tfrac{16}{3(D-2)}\phi \ , \tag{79}$$

[8] The strong $Sp(2,\mathbf{R})$ condition induces higher-order C corrections to the initial condition on $\widehat{\Phi}$ [17].

is the Einstein metric obeying

$$\widetilde{R}_{\mu\nu} + \tfrac{1}{4}(D-1)\tilde{g}_{\mu\nu} \;=\; 0 \;\;. \tag{80}$$

It is natural to expect that this result extend to higher orders, and that the end result be equivalent to some generalization of the Weyl transformation (79) at the level of master fields.

Barring this mixing problem, the Fronsdal operators at higher levels can be sorted out from the constraint on $\mathcal{F}$ in (75) noting that the $SO(D-1,2)$-covariant derivatives do not mix $SO(D-1,2)$ irreps, which allows one to consistently restrict the internal indices at level ℓ to the $SO(D-1,2)$ irrep with weight $\{2\ell+1, 2\ell+1\}$. Decomposing this irrep into Lorentz tensors and using cohomological methods [43, 17] then reveals that the curvatures are properly on-shell, although the precise form of the resulting field equations, as we have anticipated, requires a more careful elimination of the ideal gauge fields, that will be discussed in subsection 4.4.

Spectrum of the Model and Group Theoretical Interpretation

Although, as we have seen, the linearized field equations for the HS fields arising from the $\mathcal{F}$ constraint exhibit a mixing phenomenon, the physical degrees of freedom and the group theoretical interpretation of the resulting spectrum can be deduced from the mass-shell conditions (77) for the scalar field ϕ and the Weyl tensors. This is due to the fact that all local degrees of freedom enter the system via the 0-form sources. Thus, *the mode expansions of a gauge-fixed symmetric tensor and its Weyl tensor are based on the same lowest-weight space* $D(E_0; S_0)$, and simply differ in the embedding conditions for the irreducible Lorentz representation, that we label by J.

In order to determine the lowest-weight spaces that arise in the spectrum, one can thus treat AdS_D as the coset space $SO(D-1,2)/SO(D-1,1)$ and perform a standard harmonic analysis [46] of (77), expanding $C^{\{s,s\}}$ in terms of all $SO(D-1,2)$ irreps that contain the $SO(D-1,1)$ irrep $\{s,s\}$. It follows that these irreps have the lowest weight $\{E_0, S_0\}$, where $S_0 \equiv \{s_1, s_2\}$ (recall that we are suppressing the zeros, as usual) with $E_0 \geq s \geq s_1 \geq s \geq s_2 \geq 0$. Taking into account the Bianchi identity satisfied by $\Phi^{(s,s)}$, one can then show that $s_2 = 0$ [17], and therefore

$$S_0 = \{s\} \;. \tag{81}$$

Next, one can use the standard formula that relates the Laplacian ∇^2 on AdS_D acting on the $SO(D-1,2)$ irreps described above to the difference of the second order Casimir operators for the $SO(D-1,2)$ irrep $\{E_0, \{s\}\}$ and the $SO(D-1,1)$ irrep $\{s,s\}$, thus obtaining the characteristic equation

$$\frac{1}{4}\Big(\; C_2[SO(D-1,2)|E_0;\{s\}] \;-\; C_2[SO(D-1,1)|\{s,s\}] \;\Big) \;+\; \frac{1}{2}\,(s+D-3) \;=\; 0 \;. \tag{82}$$

The well-known formula for the second-order Casimir operators involved here then leads to

$$\frac{1}{4}\Big(\,[E_0(E_0-D+1)+s(s+D-3)] - [2s(s+D-3)]\,\Big) \;+\; \frac{1}{2}\,(s+D-3) \;=\; 0 \;, \tag{83}$$

with the end result that

$$E_0 \;=\; \frac{D-1}{2} \pm \left(s + \frac{D-5}{2}\right) \;=\; \begin{cases} s+D-3 \\ 2-s \end{cases} \;. \tag{84}$$

The root $E_0 = 2-s$ is ruled out by unitarity, except for $D=4$ and $s=0$, when both $E_0=2$ and $E_0=1$ are allowed. These two values correspond to Neumann and Dirichlet boundary conditions on the scalar field, respectively. Hence, the spectrum for $D \geq 5$ is given by

$$\mathcal{S}_D \;=\; \bigoplus_{s=0,2,4,\ldots} D(s+D-3, \{s\}_{D-1}) \;. \tag{85}$$

while the theory in $D = 4$ admits two possible spectra, namely $\mathcal{S}_4$ and

$$\mathcal{S}'_4 \;=\; D(2,0) \;\oplus \bigoplus_{s=2,4,\ldots} D(s+1,s) \;. \tag{86}$$

In $D = 3$ the twisted-adjoint representation is one-dimensional, and hence [34, 17, 15]

$$\mathcal{S}_3 \;=\; R \;. \tag{87}$$

As discussed in Section 2, the spectrum (85) fills indeed a unitary and irreducible representation of $ho_0(D-1,2)$ isomorphic to the symmetric product of two scalar singletons [34, 15]. The alternative 4D spectrum $\mathcal{S}'_4$ is also a unitary and irreducible representation of $ho_0(3,2)$, given by the anti-symmetric product of two spinor singletons $D(1,\frac{1}{2})$. These arise most directly in the spinor oscillator realization of $ho_0(3,2)$, often referred to as $hs(4)$ (see (15)).

4.4 Compensator Form of the Linearized Gauge-Field Equations

Leaving aside the mixing problem, and considering for simplicity the flat limit, one is thus faced with the linearized curvature constraints ($k = 0, \ldots, s-1$)[9]

$$\begin{aligned}
\mathcal{F}^{(s-1,k)}_{\mu\nu,a(s-1),b(k)} \;&\equiv\; 2\partial_{[\mu} W^{(s-1,k)}_{\nu],a(s-1),b(k)} + 2c_{s,k} W^{(s-1,k+1)}_{[\mu|,a(s-1),|\nu]b(k)} \\
&=\; \delta_{k,s-1} C^{\{s,s\}}_{[\mu|a(s-1),|\nu]b(s-1)} \;,
\end{aligned} \tag{88}$$

where $c_{s,k}$ is a constant, whose precise value is inconsequential for our purposes here, and the gauge fields $W^{(s-1,k)}_\mu$ are traceful, while the 0-form $C^{\{s,s\}}$ on the right-hand side is the *traceless* spin-s Weyl tensor. The trace of (88) in a pair of internal indices generates a homogeneous set of cohomological equations of the type considered by Dubois-Violette and Henneaux [44], and the analysis that follows is in fact a combination of the strongly projected Vasiliev equations with the results of Bekaert and Boulanger [24] on the link between the Freedman-de Wit connections and the compensator equations of [21, 22]. Let us begin by discussing some preliminaries and then turn to the cases of spin 3 and 4. These exhibit all the essential features of the general case, that will be discussed in [17].

The scheme for eliminating auxiliary fields parallels the discussion of the off-shell case. Thus, after fixing the Stückelberg-like shift symmetries, one is led to identify the independent spin-s gauge field, a totally symmetric rank-s tensor $\phi_{a(s)} \equiv \phi_{a_1\ldots a_s}$, via

$$W^{(s-1,0)}_{\mu,a(s-1)} \;=\; \phi_{a(s-1)\mu} \;, \tag{89}$$

where the hooked Young projection can be eliminated by a gauge-fixing condition. The constraints on $\mathcal{F}^{(s-1,k-1)}_{\mu\nu}$, for $k = 1, ..., s-1$, determine the *Freedman-de Wit connections*, or generalized Christoffel symbols

$$W^{(s-1,k)}_{\mu,a(s-1),b(k)} \;\equiv\; \gamma_{s,k}\; \partial^k{}_{\underbrace{b(k)\phi_{a(s-1)}}_{(s-1,k)}\mu} \;, \tag{90}$$

where the subscript $(s-1,k)$ defines the Young projection for the right-hand side and the $\gamma_{s,k}$ are constants whose actual values are immaterial for the current discussion. The remaining gauge transformations are then, effectively,

$$\delta_\Lambda \phi_{a(s)} \;=\; s\, \partial_{(a_1} \Lambda_{a_2\ldots a_s)} \;, \tag{91}$$

and are of course accompanied by gauge-preserving shift parameters $\epsilon^{(s-1,k)}$. It is important to note that the gauge-fixed frame fields $W^{(s-1,k)}$ ($k = 0, 1, ..., s-1$) are *irreducible* $SL(D)$ tensors of type (s,k), since their $(s-1,k+1)$ and $(s-1,k,1)$ projections vanish as a result of too many antisymmetrizations. On the other hand, the traces $W^{(s-1,k)}_{\mu,a(s-1),b(k-2)c}{}^c$ for $k = 2, \ldots, s-1$ are *reducible* $SL(D)$ tensors, comprising $(s,k-2)$ and $(s-1,k-1)$ projections (the remaining $(s-1,k-2,1)$ projection again vanishes due to

[9]In this section we use the notation in which $a(k)$ denotes a symmetric set of k Lorentz vector indices.

too many antisymmetrizations). In particular, the $(s-1,k-1)$ part does not contain the d'Alembertian and takes the form

$$\underbrace{W^{(s-1,k)}_{\mu,\alpha(s-1),b(k-2)}}_{(s-1,k-1)} = \underbrace{\partial^{k-1}_{b(k-1)} j_{a(s-1)}}_{(s-1,k-1)} \ , \tag{92}$$

where the constructs $j_{a(s-1)}$ are linear combinations of $\partial\cdot\phi_{a(s-1)}$ and $\partial_a\phi'_{a(s-2)}$. The above observations imply that the trace of (88) in a pair of internal indices

$$2\partial_{[\mu}W^{(s-1,k)}_{\nu],a(s-1),b(k-2)c}{}^{c} + 2c_{s,k}W^{(s-1,k+1)}_{[\mu|,a(s-1),|\nu]b(k-2)c}{}^{c} \ = \ 0 \ , \qquad k=2,\ldots,s-1 \ , \tag{93}$$

results in a homogenous cohomological problem for $W^{(s-1,k)}_{\mu,a(s-1),b(k-2)c}{}^{c}$ for $k=2,\ldots,s-1$, whose integration gives rise to the compensators as "exact" forms.

For the sake of clarity, let us begin by examining the spin-3 case. Here the relevant trace part, given by

$$W^{(2,2)}_{\mu,ab,c}{}^{c} \ = \ \frac{\gamma_{3,2}}{3}\,\mathcal{K}_{\mu,ab} \ , \qquad \text{where} \quad \mathcal{K}_{\mu,ab} \ = \ \Box\phi_{ab\mu} - 2\partial_{(a}\partial\cdot\phi_{b)\mu} + \partial_a\partial_b\phi'_\mu \ , \tag{94}$$

and where ϕ' denotes the trace of ϕ and $\gamma_{3,2}$ is a constant whose actual value plays no role in our argument, obeys the constraint

$$\partial_{[\mu}W^{(2,2)}_{\nu],ab,c}{}^{c} \ = \ 0 \ . \tag{95}$$

Hence, $\mathcal{K}$ can be related to a symmetric rank-2 tensor:

$$\mathcal{K}_{\mu,ab} \ = \ \partial_\mu\beta_{ab} \ , \tag{96}$$

where $\beta_{a,b}$ is constrained since the $(2,1)$ projection of (96), equivalent to anti-symmetrizing on μ and a, implies that

$$\partial_{[\mu}(\partial\cdot\phi_{a]b} - \partial_{a]}\phi'_b) \ = \ \partial_{[\mu}\beta_{a]b} \ , \tag{97}$$

whose solution reads

$$\beta_{ab} \ = \ \partial\cdot\phi_{ab} - 2\partial_{(a}\phi'_{b)} \ + \ 3\partial_a\partial_b\alpha \ . \tag{98}$$

Notice that the last term is a homogeneous solution parameterized by an unconstrained scalar function α and involves two derivatives. Plugging this expression for β_{ab} into the $(3,0)$ projection of (96) finally gives

$$\mathcal{F}_{abc} \ \equiv \ \Box\phi_{abc} - 3\partial_{(a}\partial\cdot\phi_{bc)} + 3\partial_{(a}\partial_b\phi'_{c)} \ = \ 3\partial^3_{abc}\alpha \ , \tag{99}$$

the flat-space version of the spin-3 compensator equation (3).

The origin of the triple gradient of the compensator can be made more transparent by an alternative derivation, that has also the virtue of generalizing more simply to higher spins. To this end, one can analyze the consequences of the invariance under the gauge transformation

$$\delta_\Lambda\phi_{abc} \ = \ 3\partial_{(a}\Lambda_{bc)} \ , \tag{100}$$

that affects the left-hand side of (96) according to

$$\delta_\Lambda\mathcal{K}_{\mu,ab} \ = \ \partial_\mu\widetilde{\Lambda}_{ab} \ , \qquad \widetilde{\Lambda}_{ab} \ = \ \Box\Lambda_{ab} - 2\partial_{(a}\partial\cdot\Lambda_{b)} + \partial^2_{ab}\Lambda' \ . \tag{101}$$

Gauge invariance thus implies that the homogeneous solution β_{ab} must transform as

$$\delta_\Lambda\beta_{ab} \ = \ \widetilde{\Lambda}_{ab} \ . \tag{102}$$

From $\delta_\Lambda\partial\cdot\phi_{ab} = \Box\Lambda_{ab} + 2\partial_{(a}\partial\cdot\Lambda_{b)}$ and $\delta_\Lambda\partial_{(a}\phi'_{b)} = \partial_a\partial_b\Lambda' + 2\partial_{(a}\partial\cdot\Lambda_{b)}$, one can see that β_{ab} is a linear combination of $\partial\cdot\phi_{ab}$, $\partial_{(a}\phi'_{b)}$ and of a term $\partial^2_{ab}\alpha$, with α an independent field transforming as

$$\delta_\Lambda\alpha \ = \ \Lambda' \ . \tag{103}$$

This in its turn implies that the symmetric part of (96) is a gauge covariant equation built from $\Box\phi_{abc}$,

$\partial_{(a}\partial\cdot\phi_{bc)}$, $\partial^2_{(ab}\phi'_{c)}$ and $\partial^3_{abc}\alpha$, and uniquely leads to the compensator equation (99).

The direct integration for spin 4 is slightly more involved, since two traced curvature constraints,

$$\partial_{[\mu}W^{(3,3)}_{\nu],a(3),bc}{}^{c} \;=\; 0\;, \tag{104}$$

$$\partial_{[\mu}W^{(3,2)}_{\nu],a(3),c}{}^{c} \;+\; c_{4,2}\,W^{(3,3)}_{[\mu|,a(3),|\nu]c}{}^{c} \;=\; 0\;, \tag{105}$$

have to be dealt with in order to obtain the compensator equation. The first constraint implies that

$$W^{(3,3)}_{\mu,a(3),bc}{}^{c} \;=\; \partial_\mu\beta^{(3,1)}_{a(3),b}\;, \tag{106}$$

where the explicit form of the Freedman-de Wit connection is

$$\frac{4}{\gamma_{4,3}}\,W^{(3,3)}_{\mu,a(3),bc}{}^{c} \;=\; \partial_b\Box\phi_{a(3)\mu}-2\partial_a\Box\phi_{a(2)b\mu}-\partial^2_{ab}\partial\cdot\phi_{a(2)\mu}-\partial^3_{a(2)b}\phi'_{b\mu}-2\partial^2_{a(2)}\partial\cdot\phi_{ab\mu}+\partial^3_{a(3)}\phi'_{b\mu}\;, \tag{107}$$

with $\gamma_{4,3}$ a constant whose actual value plays no role in our argument. The $(3,2)$ projection does not contain any d'Alembertians, and can thus be identified with the double gradient of $\mathcal{J}_{a(3)}$, a construct of $\partial\cdot\phi_{a(3)}$ and $\partial_a\phi'_{a(2)}$. It can be integrated once, with the result that

$$\beta^{(3,1)}_{a(3),b} \;=\; \underbrace{\partial_b\mathcal{J}_{a(3)}}_{(3,1)} \;+\; \underbrace{\partial^3_{a(3)}\alpha_b}_{(3,1)}\;, \tag{108}$$

where the last term, with α_b an unconstrained vector field, is the general solution of the homogeneous equation, *i.e.* its cohomologically exact part in the language of Dubois-Violette and Henneaux [44]. Eq. (108) implies that the second curvature constraint can be written in the form

$$\partial_{[\mu}\,W^{(3,2)}_{\nu],a(3),c}{}^{c} \;+\; c_{4,2}\,\partial_{[\mu|}\beta^{(3,1)}_{a(3),|\nu]} \;=\; 0\;, \tag{109}$$

and integrating this equation one finds

$$W^{(3,2)}_{\mu,a(3),c}{}^{c} \;+\; c_{4,2}\,\beta^{(3,1)}_{a(3),\mu} \;=\; \partial_\mu\beta^{(3)}_{a(3)}\;, \tag{110}$$

where the homogeneous term $\beta^{(3)}$ is constrained, since a derivative can be pulled out from the $(3,1)$ projection of the left-hand side. Thus, using

$$\underbrace{\partial^3_{a(3)}\alpha_b}_{(3,1)} \;=\; -\,\underbrace{\partial_b\,\partial^2_{a(2)}\alpha_a}_{(3,1)}\;, \tag{111}$$

one finds that

$$\beta^{(3)}_{a(3)} \;=\; \widetilde{\mathcal{J}}_{a(3)} \;-\; c_{4,2}\,\partial^2_{a(2)}\alpha_a\;, \tag{112}$$

where $\widetilde{\mathcal{J}}_{a(3)}$ is another construct of the first derivatives of ϕ, and the homogeneous solution has been absorbed into a shift of α_a by a gradient. Finally, the $(4,0)$ projection of (110) reads

$$W^{(3,2)}_{a,a(3),c}{}^{c} \;=\; \partial_a\widetilde{\mathcal{J}}_{a(3)} \;-\; c_{4,2}\partial^3_{a(3)}\alpha_a\;, \tag{113}$$

and gauge invariance implies that, up to an overall rescaling of α_a, this is the flat-space compensator equation (3) for spin 4.

In summary, a careful analysis of the Vasiliev equations shows that, if the gauge fields are left free to fluctuate and adjust themselves to constrained Weyl 0-form sources, one is led, via the results of [24], to the compensator equations of [21, 22] rather than to the conventional Fronsdal formulation.

5 Towards a finite curvature expansion

In this section we state the basic problem one is confronted with in the perturbative analysis of the strong $Sp(2,R)$-invariance condition, and discuss how finite results could be extracted from the non-linear Vasiliev equations in this setting. These observations rest on properties of the $\star$-products of the singular projector and other related non-polynomial objects, that we shall illustrate with reference to a simpler but similar case, the strong $U(1)$ condition that plays a role in the 5D vector-like construction [4,20] based on spinor oscillators [13]. Whereas these novel possibilities give a concrete hope that a finite construction be at reach, a final word on the issue can not forego a better understanding of the interactions in the actual $Sp(2,R)$ setting. We intend to return to these points in [17].

5.1 On the Structure of the Interactions

In the previous section, we found that the strong $Sp(2,R)$ condition (55) led to a *linearized* zero-form master field Φ with an expansion involving non-polynomial dressing functions $F(N;K^2)$, according to (72). In particular, the master-field projector M, introduced in (66) and such that $\Phi = M \star C$,

$$M(K^2) \;=\; F(0;K^2)\;, \tag{114}$$

is the dressing function of the scalar field ϕ component of Φ. The dressing functions are proportional to $J_\nu(\xi)/(\xi/2)^\nu$, where J_ν is a Bessel function of order $\nu = (N+D-2)/2$ and of argument $\xi = 4\sqrt{K^2}$, and hence satisfy the Laplace-type differential equation

$$\left(\xi\,\frac{d^2}{d\xi^2} + (2\nu+1)\,\frac{d}{d\xi} + \xi\right) F(N;\xi^2/16) \;=\; 0\;. \tag{115}$$

The solution of this equation that is analytic at the origin can be given the real integral representation

$$F(N;K^2) \;=\; \mathcal{N}_\nu \int_{-1}^{1} ds\;(1-s^2)^{\nu-\frac{1}{2}}\;\cos\left(4\sqrt{K^2}\,s\right)\;, \tag{116}$$

where the normalization, fixed by $F(N;0)=1$, is given by

$$\mathcal{N}_\nu \;=\; \frac{1}{B\left(\nu+\frac{1}{2},\frac{1}{2}\right)} \;=\; \frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\;, \tag{117}$$

with B the Euler B-function. Alternatively, in terms of the variable $z = K^2$, the differential equation takes the form

$$\left(\frac{z}{4}\,\frac{d^2}{dz^2} + \frac{\nu+1}{4}\,\frac{d}{dz} + 1\right) F(N;z) \;=\; 0\;, \tag{118}$$

which is also of the Laplace type, and leads directly to the Cauchy integral representation

$$F(N;K^2) \;=\; \Gamma(\nu+1)\oint_\gamma \frac{dt}{2\pi i}\,\frac{\exp\left(t-\frac{4K^2}{t}\right)}{t^{\nu+1}}\;, \tag{119}$$

where the Hankel contour γ encircles the negative real axis and the origin.

In building the perturbative expansion of the strongly projected zero form, one also encounters an additional non-polynomial object, that we shall denote by $G(K^2)$ and plays the role of a $\star$-*inverse* of the $Sp(2,R)$ Casimir operator. It is the solution of

$$K^I \star K_I \star G \star H \;=\; H\;, \tag{120}$$

where H belongs to a certain class of functions not containing $M(K^2)$ [17]. This element instead obeys

$$K^I \star K_I \star G \star M \;=\; 0\;, \tag{121}$$

in accordance with the associativity of the $\star$-product algebra.

Dealing with the above non-polynomial operators, all of which can be given contour-integral representations, requires pushing the $\star$-product algebra beyond its standard range of applicability. The integral representations can be turned into forms containing exponents linear in K_I, suitable for $\star$-product compositions, at the price of further parametric integrals, but one then discovers the presence of a one-dimensional *curve of singularities* in the parametric planes, leading to an actual *divergence* in $M \star M$ [34].

Let us consider in more detail the structure of the interactions. At the n-th order they take the form

$$\widehat{\mathcal{O}}_1 \star M \star \widehat{\mathcal{O}}_2 \star M \star \widehat{\mathcal{O}}_3 \star \cdots \star M \star \widehat{\mathcal{O}}_{n+1} \ , \tag{122}$$

where the $\widehat{\mathcal{O}}_s$, $s = 1, \dots, n+1$ are built from components of the master fields A_μ and C defined in (74), the element $\kappa(t)$ defined in (27), and the $\star$-inverse functions $G(K^2)$. One possible strategy for evaluating this expression would be to group the projectors together, rewriting it as

$$\widehat{\mathcal{O}}_1 \star \underbrace{M \star M \star \cdots \star M}_{n \text{ times}} \star [\widehat{\mathcal{O}}_2]_0 \star [\widehat{\mathcal{O}}_3]_0 \star \cdots \star [\mathcal{O}_n]_0 \star \widehat{\mathcal{O}}_{n+1} \ , \tag{123}$$

where $[\widehat{\mathcal{O}}]_0$ denotes the $Sp(2,R)$-singlet, or neutral, projection[10] of $\widehat{\mathcal{O}}$. This procedure clearly leads to divergent compositions that appear to spoil the curvature expansion. However, (123) is only an intermediate step in the evaluation of (122), and the singularities may well cancel in the final Weyl-ordered form of (122). Therefore, one should exploit the associativity of the $\star$-product algebra to find a strategy for evaluating (122) that may avoid intermediate singular expressions altogether. To achieve this, one may first $\star$-multiply the projectors with adjacent operators $\widehat{\mathcal{O}}_s$, and then consider further compositions of the resulting constructs. It is quite possible that this rearrangement of the order of compositions result in interactions that are actually completely free of divergences, as we shall discuss further below.

In principle, one may instead consider a modified curvature expansion scheme, based on an additional $Sp(2,R)$ projector, which we shall denote by $\Delta(K^2)$, that is of distributional nature and has finite compositions with both the analytic projector and itself [17]. We shall return to a somewhat more detailed discussion of possible expansion schemes in Section 5.3, while we next turn to the analysis of a simpler $U(1)$ analog.

5.2 A Simpler $U(1)$ Analog

In this section we examine a strong $U(1)$ projection that is simpler than the actual $Sp(2,R)$ case but exhibits similar features. The key simplification is that the corresponding non-polynomial objects admit one-dimensional parametric representations, directly suitable for $\star$-product compositions, that only contain *isolated singularities* in the parameter planes. Drawing on these results, in Section 5.3 we shall finally discuss two possible prescriptions for the curvature expansion.

Analytic $U(1)$ Projector

The $Sp(2,R)$ dressing functions $F(N;K^2)$ are closely related to others occurring in the 5D and 7D constructions of [4] and [19]. These are based on bosonic Dirac-spinor oscillators [13] y_α and $\bar{y}^\beta$ obeying

$$y_\alpha \star \bar{y}^\beta \ = \ y_\alpha \bar{y}^\beta + \delta_\alpha^\beta \ , \tag{124}$$

and providing realizations of the minimal bosonic higher-spin algebras $ho_0(4,2)$ and $ho_0(6,2)$, at times referred to as $hs(2,2)$ and $hs(8^*)$, via on-shell master fields subject to $U(1)$ and $SU(2)$ conditions,

[10] In general, any polynomial in oscillators can be expressed as a finite sum of $(2k+1)$-plets. The neutral part is the singlet, $k=0$, component. A properly refined form of eq. (123) applies also to more general structures arising in the Φ-expansion, where the inserted operators can carry a number of doublet indices.

respectively. In particular, the 5D $U(1)$ generator is given by [4]

$$x \ = \ \bar{y}y \ . \tag{125}$$

One can show that for $2n$-dimensional Dirac spinor oscillators

$$x \star f(x) \ = \ \left(x \ - \ 2\,n\,\frac{d}{dx} \ - \ x\,\frac{d^2}{dx^2} \right) f(x) \ , \tag{126}$$

where $n = 2$ in $D = 5$. The strong $U(1)$-invariance condition defining the 5D Weyl zero form [4] can be formally solved introducing a projector $m(x)$ analytic at the origin and such that [20]

$$x \star m(x) \ = \ 0 \ . \tag{127}$$

The analytic projector m belongs to a class of $U(1)$ dressing functions $f(N;x)$, that arise in the component-field expansion of the linearized Weyl master zero form [4], and which obey

$$\left(x\,\frac{d^2}{dx^2} \ + \ (2\nu+1)\,\frac{d}{dx} \ - \ x \right) f(N;x) \ = \ 0 \ , \tag{128}$$

with $\nu = n + N - \frac{1}{2}$, so that

$$m(x) \ = \ f(0;x) \ . \tag{129}$$

The solution of (128) that is analytic at the origin can be expressed in terms of the modified Bessel function of index ν and argument x as $I_\nu(x)/x^\nu$, and admits the real integral representation [45]

$$f(N;x) \ = \ \mathcal{N}_\nu \int_{-1}^{1} ds \ (1-s^2)^{\nu-\frac{1}{2}} \ e^{sx} \ , \tag{130}$$

where the normalization $\mathcal{N}_\nu$, determined by the condition that $f(N;0) = 1$, is as in (117).

In order to compose m with other objects, it is useful to recall the $\star$-product formula

$$e^{sx} \star e^{s'x} \ = \ \frac{e^{t(s,s')x}}{(1+ss')^{2n}} \ , \quad \text{with} \ \ t(s,s') \ = \ \frac{s+s'}{1+ss'} \ , \tag{131}$$

that holds for arbitrary values of $s, s' \in \mathbf{C}$ such that $1 + ss' \neq 0$, and to note that the projective transformation $t(s,s')$ has the form of the relativistic addition of velocities, so that $t(\pm 1, s') = t(s, \pm 1) = \pm 1$. This can be used to show that, for any complex number $\lambda \neq \pm 1$,

$$e^{\lambda x} \star m(x) \ = \ \frac{1}{(1-\lambda^2)^n} \ m(x) \ , \tag{132}$$

and hence that the $\star$-product of a pair of analytic projectors results in the singular expression

$$m(x) \ \star \ m(x) \ = \ \mathcal{N} \, m(x) \int_{-1}^{1} \frac{ds}{1-s^2} \ , \tag{133}$$

where $\mathcal{N} \equiv \mathcal{N}_{n-\frac{1}{2}}$. The logarithmic nature of the singularity is consistent with power counting in the double-integral expression

$$m(x) \ \star \ m(x) \ = \ \mathcal{N}^2 \int_{-1}^{1} ds \int_{-1}^{1} ds' \ \frac{(1-s^2)^{n-1}(1-s'^2)^{n-1}}{(1+ss')^{2n}} \ e^{t(s,s')x} \tag{134}$$

where the integrand diverges as ϵ^{-2} as $s \sim 1 - \epsilon$ and $s' \sim -1 + \epsilon$.

As discussed in Section 5.1, the above singularity need not arise in the curvature expansion of a HS theory based on the spinor oscillators. Indeed, if one first composes the m's with other operators,

expands in terms of dressing functions, and continues by composing these, one encounters

$$f(N;x) \;\star\; f(N';x) \;=\; \mathcal{N}_\nu\, \mathcal{N}_{\nu'} \sum_{k=0}^{N'} \binom{2N'}{2k} B\left(k+\frac{1}{2}, N+N'\right) I_{N,N';k}(x) \;, \tag{135}$$

with $\nu = n+N-\frac{1}{2}$, $\nu' = n+N'-\frac{1}{2}$ and

$$I_{N,N';k}(x) \;=\; \int_{-1}^{1} dt\; t^{2k}(1-t^2)^{\nu'-\frac{1}{2}}\, {}_2F_1\left(k+\frac{1}{2}, 2N'; N+N'+k+\frac{1}{2}; t^2\right)\; e^{tx} \;, \tag{136}$$

where ${}_2F_1$ is the hypergeometric function. This type of expression is free of divergences for $N+N' \geq 1$, while it is apparently equal to the singular expression $\mathcal{N}\,\Gamma(0)\, m(x)$ if $N=N'=0$, but in fact, using ${}_2F_1(0,\beta;\gamma;z)=1$, one can see that

$$f(N;x) \;\star\; m(x) \;=\; \mathcal{N}_\nu\; B\left(N, \frac{1}{2}\right)\; m(x) \;. \tag{137}$$

More generally, multiple compositions $f(N_1;x)\star\cdots\star f(N_n;x)$ are finite for $N_1+\cdots+N_n \geq 1$, as can be seen by power counting. Therefore, the finiteness of the interactions is guaranteed provided the case $N_1=\cdots=N_n=0$ never presents itself, a relatively mild condition that could well hold for the interactions [11].

It is interesting to examine more closely the nature of the singularity in (133).

Problems with Regularization of the Singular Projector

The singularities of (133) can be removed by a cutoff procedure, at the price of violating $U(1)$ invariance. For instance, given the regularization

$$m_\epsilon(x) \;=\; \mathcal{N}\int_{-1+\epsilon}^{1-\epsilon} ds\; e^{sx}\; (1-s^2)^{n-1} \;, \tag{138}$$

one can use (126) to show that

$$x \star m_\epsilon(x) \;=\; 2\,\mathcal{N}\,[1-(1-\epsilon)^2]^n\;\; \sinh[(1-\epsilon)x] \;, \tag{139}$$

where $\sinh[(1-\epsilon)x]$ belongs to the ideal, since $\sinh[\lambda x]\star m(x) = 0$ for all λ . However, while the resulting violation may naively appear to be small, it is actually sizeable, since $\sinh[(1-\epsilon)x]$ and $m_\epsilon(x)$ have a very singular composition, with the end result that an *anomalous finite violation* of $U(1)$ invariance can emerge in more complicated expressions, so that for instance

$$x\star m_\epsilon(x)\star m_\epsilon(x) \;=\; \mathcal{N}^2 \int_0^1 ds\; (1-s^2)^{n-1}\sinh(sx) \;+\; \text{evanescent} \;. \tag{140}$$

This anomaly can equivalently be computed first composing

$$m_\epsilon(x)\star m_\epsilon(x) \;=\; \mathcal{N}^2 \int_0^{t(1-\epsilon,1-\epsilon)} du\; \int_{-u}^{u} dt\; (1-t^2)^{n-1}\; e^{tx} \;, \tag{141}$$

and then expanding in ϵ, which yields

$$m_\epsilon(x)\star m_\epsilon(x) \;=\; \mathcal{N}\log\left(\frac{2-\epsilon}{\epsilon}\right)\; m(x) \;+\; A^{(2)}(x) \;+\; \text{evanescent} \;, \tag{142}$$

[11]This corresponds, roughly speaking, to constraints on scalar-field *non-derivative* self interactions, which might be related to the fact that all such couplings actually vanish in the 4D spinor-formulation as found in [42].

where the divergent part is proportional to $m(x)$, while the finite, anomalous, part

$$A^{(2)}(x) \;=\; \mathcal{N}^2 \int_0^1 du\, \log\left(\frac{1+u}{1-u}\right)\, (1-u^2)^{n-1}\sinh ux\;, \tag{143}$$

belongs to the ideal. One can verify that its composition with x indeed agrees with the right-hand side of (141), in compliance with associativity. Continuing in this fashion, one would find that higher products of $m_\epsilon(x)$ with itself keep producing anomalous finite terms together with logarithmic singularities, all of which are proportional to $m(x)$, together with tails of evanescent terms that vanish like powers or powers times logarithms when the cutoff is removed [17]:

$$\underbrace{m_\epsilon(x)\star\cdots m_\epsilon(x)}_{p\ \text{times}} \;=\; \sum_{l=-p+1}^{-1}\left(\log\frac{1}{\epsilon}\right)^l c_l^{(p)} m(x) \;+\; F^{(p)}(x) + \sum_{k\geq 1, l\geq 0} \epsilon^k \left(\log\frac{1}{\epsilon}\right)^l E_{k,l}^{(p)}(x)\;, \tag{144}$$

with $F^{(p)}(x) = c^{(0)}m(x) + A^{(p)}(x)$, where $A^{(p)}(x)$ represents the anomaly.

We would like to stress that the integral representations should be treated with some care, as can be illustrated considering

$$m_\epsilon(x)\star m(x) \;=\; \mathcal{N}\,\log\left(\frac{2-\epsilon}{\epsilon}\right)\, m(x)\;, \tag{145}$$

and its generalization

$$m(x)\star\underbrace{m_\epsilon(x)\star\cdots\star m_\epsilon(x)}_{p\ \text{times}} \;=\; \left(\mathcal{N}\,\log\frac{2-\epsilon}{\epsilon}\right)^p\, m(x)\;. \tag{146}$$

Comparing (146) with (142), it should be clear that one should perform the parametric integrals prior to expanding in ϵ, since one would otherwise encounter ill-defined compositions of "bare" m's.

In order to appreciate the meaning of the anomaly, let us consider a tentative Cartan integrable system[12] on the direct product of spacetime with an internal noncommutative z-space, containing a strongly $U(1)$-projected 0-form master field $\widehat{\Phi}$ with a formal perturbative expansion in $m\star C$. At the n-th order, one could use the $U(1)$ analog of the rearrangement in (123) to bring all projectors together, and then attempt to regulate the resulting divergent $\star$-product compositions, for instance replacing each m by an m_ϵ. This induces a violation of the constraints in z-space. Therefore, in order to preserve the Cartan integrability in spacetime, one could use *unrestricted* z-expansions for the full master fields, at the price of introducing "spurious" space-time fields that one would have to remove by some form of consistent truncation. To analyze this, one first observes that the structure of the ϵ-dependence in (144) implies that, the Cartan integrability condition satisfied by the finite part of the regulated interactions can not contain any contributions from singular times evanescent terms. Thus, the finite part in itself constitutes a consistent set of interactions, albeit for all the space-time fields, including the spurious ones.

In the absence of anomalies it would be consistent to set to zero all the spurious fields, thus obtaining a well-defined theory for the original space-time master fields. However, as the earlier calculations show, these anomalies arise inevitably and jeopardize these kinds of constructions, in that they must cancel in the final form of the interactions, along lines similar to those discussed in Section 5.1 and indicated below (135).

Distributional $\star$-Inverse Function and Normalizable Projector

As we have seen, the $U(1)$ projection condition $x\star m(x) = 0$ corresponds to the Laplace-type differential equation determined by (126), which admits two solutions that are ordinary functions of x, so that the

[12]These considerations would apply, in their spirit, to the 5D spinor construction, whose completion into an integrable non-linear system is still to be obtained.

singular projector $m(x)$ is the unique one that is also analytic at $x = 0$. Interestingly, there also exist solutions that are distributions in x. Their Laplace transforms involve e^{sx} with an imaginary parameter s, which improves their $\star$-product composition properties, and in particular allows for a normalizable projector. A related distribution, with similar properties, provides a $U(1)$ analog of the $\star$-inverse of the $Sp(2,R)$ Casimir defined in (120).

To describe these objects, it is convenient to observe that, if γ is a path in the complex s-plane, the function

$$p_\gamma(x) \ = \ \int_\gamma ds \ (1-s^2)^{n-1} \ e^{sx} \tag{147}$$

obeys

$$x \star p_\gamma \ = \ \left[(1-s^2)^n \ e^{sx}\right]_{\partial\gamma} \ . \tag{148}$$

For instance, the projector $m(x) \sim I_\nu(x)/x^\nu$, with $\nu = n - \frac{1}{2}$, obtains taking for γ the unit interval, while including boundaries also at $\pm\infty$ yields in general linear combinations of $I_\nu(x)/x^\nu$ and $K_\nu(x)/x^\nu$. The latter is not analytic at $x = 0$, however, and as such it would not seem to play any role in the dressing of the linearized zero-form master field. An additional possibility is provided by the standard representation of Dirac's δ function,

$$\delta(x) \ = \ \int_{-\infty}^{\infty} \frac{dt}{2\pi} \ e^{itx-\epsilon t^2} \tag{149}$$

with $\epsilon \to 0^+$, which suggests that boundaries at $\pm i\infty$ might give rise to projectors that are distributions in x. To examine this more carefully, let us consider

$$d(x) \ = \ \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \ (1-s^2)^{n-1} \ e^{xs+\epsilon s^2} \ . \tag{150}$$

Treating $d(x)$ as a distribution acting on test functions $f(x)$ such that $f^{(2k)}(0)$ fall off fast enough with k, one can indeed expand it as

$$d(x) \ = \ \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \delta^{(2k)}(x) \ . \tag{151}$$

One can now verify that $x \star d(x) = 0$ holds in the sense that $\left(x - 2n\,\frac{d}{dx} - x\frac{d^2}{dx^2}\right) d(x)$ vanishes when smeared against test functions,

$$\begin{aligned} &\int_{-\infty}^{\infty} dx\, f(x)\,(x \star d(x)) \qquad\qquad (152)\\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \ \left[(xf(x))^{(2k)} + 2n\, f^{(2k+1)}(x) - (xf(x))^{(2k+2)}\right]\Big|_{x=0} \ = \ 0 \ . \end{aligned}$$

We can now look more closely at the composition properties of $d(x)$, beginning with

$$d(x) \star m(x) \ = \ \mathcal{N} \int_{-i\infty}^{i\infty} \frac{ds}{\pi i} \int_{-1}^{1} ds' \ \frac{(1-s^2)^{n-1}(1-s'^2)^{n-1}}{(1+ss')^{2n}} \ e^{t(s,s')x+\epsilon s^2} \ , \tag{153}$$

that can be obtained from (131). Notice that the denominator is no longer singular, since $1 + ss' \neq 0$ for $(s,s') \in iR \times [-1,1]$, and after a projective change of integration variable, $t(s,s') \to t$ at fixed s, one then finds

$$d(x) \star m(x) \ = \ \mathcal{N} \int_{-i\infty}^{i\infty} \frac{ds \ e^{\epsilon s^2}}{\pi i(1-s^2)} \ \int_{\gamma_s} dt \ (1-t^2)^{n-1} \ e^{tx} \ . \tag{154}$$

The original real contour has been mapped to γ_s, a circular arc from $t = -1$ to $t = +1$ crossing the imaginary axis at $t = s$, but can be deformed back to the real interval $[-1,1]$ using Cauchy's theorem,

so that

$$d(x) \star m(x) \;=\; m(x) \left(\int_{-i\infty}^{i\infty} \frac{ds\ e^{\epsilon s^2}}{\pi i (1-s^2)} \right) , \tag{155}$$

to be compared with (133). The integral over s is now convergent, and is actually equal to 1, and therefore

$$d(x) \star m(x) \;=\; m(x)\ . \tag{156}$$

The iteration of this formula then yields

$$\underbrace{d(x) \star \cdots \star d(x)}_{p \text{ times}} \;\star\; m(x) \;=\; m(x)\ . \tag{157}$$

This indicates that $d(x)$ is in fact *normalizable*, that is [17]

$$d(x) \star d(x) \;=\; d(x)\ , \tag{158}$$

as can be seen by a direct computation using a prescription in which the singularity at $ss'+1=0$ is avoided by small horizontal displacements of the contour that are eventually sent to zero, as in eqs. (167) and (168) below.

Let us next turn to the $U(1)$ analog of the $\star$-inverse of the $Sp(2,R)$ Casimir. This element, which we shall refer to as $g(x)$, is defined by

i) the inverse property

$$x \star g(x) \;\star\; h(x) \;=\; h(x)\ , \tag{159}$$

for an arbitrary ideal function $h(x)$. This milder form is the counterpart of the condition for a distributional solution d in (152).

ii) the twisted reality condition

$$(g(x))^{\dagger} \;=\; \sigma\, g(-x)\ , \tag{160}$$

where σ can be either $+1$ or -1;

iii) the orthogonality relation

$$m(x) \;\star\; g(x) \;=\; 0\ . \tag{161}$$

The two first conditions suffice to ensure that, given a linearized master 0-form obeying $x \star \Phi = 0$ and $\tau(\Phi) = \Phi^{\dagger} = \pi(\Phi)$ [4], one can construct a full master 0-form $\widehat{\Phi}$ that satisfies

1) a strong $U(1)$-invariance condition

$$\widehat{x} \;\star\; \widehat{\Phi} \;=\; 0\ , \tag{162}$$

where $\widehat{x}$ is a full (non-linear) version of the $U(1)$ generator, with a Φ-expansion given by

$$\widehat{x} \;=\; x + \sum_{n=1}^{\infty} \widehat{x}_{(n)}\ ; \tag{163}$$

2) the twisted τ and reality conditions

$$\tau(\widehat{\Phi}) \;=\; \widehat{\Phi}^{\dagger} \;=\; \pi(\widehat{\Phi})\ . \tag{164}$$

The perturbative expression for the full zero form is then given by

$$\widehat{\Phi} \;=\; (g(x) \star \widehat{x})^{-1} \star \Phi \star (\pi(\widehat{x}) \star g(x))^{-1}\ , \tag{165}$$

provided that (159) applies to $g(x) \star x \star \widehat{\Phi}_{(n)}$, in which case we note that the inverse elements in the above formula can be expanded in a geometric series using $g(x) \star \widehat{x} = 1 + \sum_{n=1}^{\infty} g \star \widehat{x}_{(n)}$.

To solve the conditions on $g(x)$, one can first use $\sinh(sx) \star m(x) = 0$ which follows from (132), to arrange that $g(x) \star m(x) = 0$ by writing $g(x) = \int_{\gamma} ds \sinh(sx) q(s)$. Using (148), a formal solution to the

condition (159) is then provided by

$$g(x) \;=\; \left[\alpha\int_{-i\infty}^{0} +(1-\alpha)\int_{-1}^{0}\right] ds\ (1-s^2)^{n-1}\ \sinh(sx)\ e^{\epsilon s^2}\ , \tag{166}$$

where $\epsilon \to 0^+$, α is a constant (to be fixed below), and we have assumed that the boundary terms in (148) at $\pm i\infty$ drop out when used in (159). These boundary terms play a role, however, in verifying associativity in $x\star g(x)\star m(x) = 0$. Assuming that it is legitimate to set $\epsilon = 0$ during these manipulations, it follows that $x \star g(x) \star m(x) = (1-\alpha)m(x)$ which fixes $\alpha = 1$.

In summary, we have found that distributions can be used to construct a normalizable projector $d(x)$ as well as a $\star$-inverse $g(x)$ of x, given by

$$d(x) \;=\; \left(\int_{-i\infty+\eta}^{0} + \int_{0}^{i\infty-\eta}\right)\frac{ds}{\pi i}\ (1-s^2)^{n-1}\ e^{sx+\epsilon s^2}\ , \tag{167}$$

$$g(x) \;=\; \frac{1}{2}\left(\int_{-i\infty+\eta}^{0} + \int_{i\infty-\eta}^{0}\right) ds\ (1-s^2)^{n-1}\ e^{sx+\epsilon s^2}\ , \tag{168}$$

with $\epsilon, \eta \to 0^+$, where η is a prescription for avoiding singularities in $d(x)\star d(x)$ and $g(x)\star g(x)$. These distributional objects can be expanded in terms of elementary distributions using the standard result

$$\int_0^\infty dt\, e^{ixt} \;=\; \pi\,\delta(x) \;+\; i\,PP\left(\frac{1}{x}\right)\ , \tag{169}$$

with PP the principal part. Furthermore,

$$\tau(d(x)) \;=\; \pi(\delta(x)) \;=\; \delta(x)\ , \qquad \pi(g(x)) \;=\; \tau(g(x)) \;=\; -g(x)\ , \tag{170}$$

and both elements are hermitian, *i.e.*

$$(d(x))^\dagger \;=\; d(x)\ , \qquad (g(x))^\dagger \;=\; g(x)\ , \tag{171}$$

so that $\sigma = -1$ in (iii).

Finally, the generalization of $m(x)\star g(x) = 0$ to arbitrary $U(1)$ dressing functions reads

$$f(N;x)\ \star\ g(x) \;=\; \frac{i\mathcal{N}_\nu}{2}\sum_{k=0}^{N-1}(-1)^k\binom{2N}{2k+1}B(k+1,N-k)I_{N;k}(x)\ , \tag{172}$$

where $\nu = n + N - \frac{1}{2}$ and

$$I_{N;k}(x) \;=\; \int_{-1+i\delta}^{1+i\delta} dt\ t^{2k+1}\ (1-t^2)^{\nu-\frac{1}{2}}\ {}_2F_1(k+1,2N;N+1;1-t^2)\ e^{tx}\ , \tag{173}$$

where $\delta \to 0^+$ is a prescription for how to encircle poles at $t = 0$, and we note that there are no inverse powers of x in this expression.

5.3 Proposals for Finite Curvature Expansion Schemes

We would like to conclude by summarizing our current understanding of how a finite curvature expansion could be obtained from the Vasiliev equations (56) supplemented with the strong $Sp(2,R)$ projection condition (55). The material presently at our disposal suggests two plausible such schemes, that we have partly anticipated, and we shall refer to as minimal and modified. We hope to report conclusively on the fate of these schemes in [17].

Minimal Expansion Scheme

In this scheme, which is the most natural one, one solves the internal constraints (39) and the strong projection condition (55) by an expansion in the linearized Weyl zero form $\Phi = M(K^2)\star C$. This object can be written using dressing functions $f(N;K^2)$, as in (72). We anticipate that the $\star$-products of the $Sp(2,R)$ dressing functions obey an analog of (135). The expansion also requires a $\star$-inverse function $G(K^2)$ obeying (120) and a suitable set of boundary conditions, analogous to those discussed in the $U(1)$ case. We expect this $\star$-inverse to be also distributional and to satisfy an $Sp(2,R)$ analog of (172).

Let us consider an n-th order interaction of the form (122). Since the $\widehat{\mathcal{O}}_s$ are either arbitrary polynomials or their $\star$-products with a $\star$-inverse function, the products $M\star\widehat{\mathcal{O}}_s$ yield either dressing functions or their products with the $\star$-inverse function, respectively. Here we note that, anticipating an $Sp(2,R)$ analog of (172), all parametric integrals associated with (distributional) $\star$-inverse functions would not give inverse powers of K^2. The resulting form of (122) could be written as a multiple integral over a set of parameters $s_i\in[-1,1]$ where the integrand contains factors $(1-s_i^2)^{\nu_i-\frac{1}{2}}$, where $\nu_i\geq(D-2)/2$, and a set $\star$-products involving exponentials of the form $e^{s_i x}$. These $\star$-products give rise to divergent powers of $(1+s_is_j)$, and the idea is that all of these would be cancelled by the remaining part of the integrand, resulting in a well-defined interaction devoid of singularities.

As mentioned earlier, however, while the singularities in the $U(1)$ case are isolated points in the parametric planes, in the $Sp(2,R)$ case one has to deal with lines of singularities, and this requires further attention before coming to a definite conclusion.

Modified Expansion Scheme

In this scheme, which is less natural than the previous one, albeit still a logical possibility, one first solves the internal constraints (39) and the strong projection condition (55) by an expansion in $\Phi=\Delta(K^2)\star C$, where the distributional $Sp(2,R)$ projector obeys $K_I\star\Delta(K^2)=0$ and is given by [17]

$$\Delta(K^2) \;=\; \int_{-\infty}^{\infty}\frac{ds}{2\pi}\,(1-s^2)^{\frac{D-3}{2}}\;\cos(4\sqrt{K^2}s)\;. \tag{174}$$

Starting from the above *distributional master fields*, and using the rearrangement in (123) to bring all Δ's together, one can define *analytic master fields* $\widehat{A}'_\mu[A,\Phi]$, $\widehat{A}'_i[\Phi]$ and $\widehat{\Phi}'[\Phi]$ by replacing the resulting single Δ by an M. The so constructed analytical master fields obey

$$\widehat{F}' \;\equiv\; d\widehat{A}'+\widehat{A}\star\widehat{A}' \;=\; \frac{i}{2}dZ^i\wedge dZ_i\;\widehat{\Phi}'\star\kappa\;, \tag{175}$$

$$\widehat{D}\Phi' \;\equiv\; d\widehat{\Phi}'+[\widehat{A},\widehat{\Phi}']_\pi \;=\; 0\;, \tag{176}$$

where it is immaterial which of the two master fields in each of the covariantizations that is primed, since the rearrangement in (123) implies that, schematically,

$$\widehat{U}^{(n)}\star\widehat{V}'^{(p)} \;=\; \widehat{U}'^{(n)}\star\widehat{V}^{(p)}\;. \tag{177}$$

Following steps similar to those outlined below eq. (41) [17], one can show the perturbative integrability of the analytic constraints, eqs. (175)-(176). As a consequence, the evaluation of (175) and (176) at $Z=0$, that is

$$\widehat{F'}_{\mu\nu}|_{Z=0} \;=\; 0\;,\qquad \widehat{D}_\mu\widehat{\Phi}'|_{Z=0} \;=\; 0\;, \tag{178}$$

give a set of integrable constraints on spacetime that describe a perturbative expansion of the full field equations.

The main difference between the minimal and modified schemes is that the latter involves additional parametric integrals associated with representations of distributional projectors. This might result in undesirable negative powers of K^2 (see (169)). Therefore, we are presently inclined to believe that the minimal scheme will be the one that ultimately leads to a finite curvature expansion of the D-dimensional

Vasiliev equations with supplementary $Sp(2, R)$ invariance conditions.

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Nonlinear Higher Spin Theories in Various Dimensions

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Abstract. In this article, an introduction to the nonlinear equations for completely symmetric bosonic higher spin gauge fields in anti de Sitter space of any dimension is provided. To make the presentation self-contained we explain in detail some related issues such as the MacDowell-Mansouri-Stelle-West formulation of gravity, unfolded formulation of dynamical systems in terms of free differential algebras and Young tableaux symmetry properties in terms of Howe dual algebras.

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1 Introduction

The Coleman-Mandula theorem [1] and its generalization [2] strongly restrict the S-matrix symmetries of a nontrivial (*i.e.* interacting) relativistic field theory in flat space-time. More precisely, the extension of the space-time symmetry algebra is at most the (semi)direct sum of a (super)conformal algebra and an internal symmetry algebra spanned by elements that commute with the generators of the Poincaré algebra. Ruling out higher symmetries via these theorems, one rules out higher spin (HS) gauge fields associated with them, allowing in practice only gauge fields of low spins (*i.e.* $s \leqslant 2$). However, as will be reviewed here, going beyond some assumptions of these no-go theorems allows to overcome both restrictions, on higher spins (*i.e.* $s > 2$) and on space-time symmetry extensions.

By now, HS gauge fields are pretty well understood at the free field level. Therefore, the main open problem in this topic is to find proper nonAbelian HS gauge symmetries which extend the space-time symmetries. These symmetries can possibly mix fields of different spins, as supersymmetry does. Even though one may never find HS particles in accelerators, nonAbelian HS symmetries might lead us to a better understanding of the true symmetries of unification models. From the supergravity perspective, the theories with HS fields may have more than 32 supercharges and may live in dimensions higher than 11. From the superstring perspective, several arguments support the conjecture [3] that the Stueckelberg symmetries of massive HS string excitations result from a spontaneous breaking of some HS gauge symmetries. In this picture, tensile string theory appears as a spontaneously broken phase of a more symmetric phase with only massless fields. In that case, superstrings should exhibit higher symmetries in the high-energy limit as was argued long ago by Gross [4]. A more recent argument came from the AdS/CFT side after it was realized [5, 6, 7, 8] that HS symmetries should be unbroken in the Sundborg–Witten limit

$$\lambda = g^2_{YM} N = \Big(\frac{R^2_{AdS}}{\alpha'}\Big)^2 \longrightarrow 0\,,$$

because the boundary conformal theory becomes free. A dual string theory in the highly curved AdS space-time is therefore expected to be a HS theory (see also [9, 10] and refs therein for recent developments).

One way to provide an explicit solution of the nonAbelian HS gauge symmetry problem is by constructing a consistent nonlinear theory of massless HS fields. For several decades, a lot of efforts has been put into this direction although, from the very beginning, this line of research faced several difficulties. The first explicit attempts to introduce interactions between HS gauge fields and gravity encountered severe problems [11]. However, some positive results [12] were later obtained in flat space-time on the existence of consistent vertices for HS gauge fields interacting with each other, but not with gravity.

Seventeen years ago, the problem of consistent HS gravitational interactions was partially solved in four dimensions [13]. In order to achieve this result, the following conditions of the no-go theorems [1, 2] were relaxed:

(1) the theory is formulated around a flat background.

(2) the spectrum contains a finite number of HS fields.

The nonlinear HS theory in four dimensions was shown to be consistent up to cubic order at the action level [13] and, later, at all orders at the level of equations of motion [14, 15]. The second part of these results was recently extended to arbitrary space-time dimensions [16]. The nonlinear HS theory exhibits some rather unusual properties of HS gauge fields:

(1') the theory is perturbed around an $(A)dS$ background and does not admit a flat limit as long as HS symmetries are unbroken.

(2') the allowed spectra contain infinite towers of HS fields and do not admit a consistent finite truncation with $s > 2$ fields.

(3') the vertices have higher-derivative terms (that is to say, the higher derivatives appear in HS interactions - not at the free field level).

The properties (2') and (3') were also observed by the authors of [12] for HS gauge fields. Though unusual, these properties are familiar to high-energy theorists. The property (1') is verified by gauged

supergravities with charged gravitinos [17, 18]. The property (2') plays an important role in the consistency of string theory. The property (3') is also shared by Witten's string field theory [19].

An argument in favor of an AdS background is that the S-matrix theorems [1, 2] do not apply since there is no well-defined S-matrix in AdS space-time [20]. The AdS geometry plays a key role in the nonlinear theory because cubic higher derivative terms are added to the free Lagrangian, requiring a nonvanishing cosmological constant Λ. These cubic vertices are schematically of the form

$$\mathcal{L}^{int} = \sum_{n,p} \Lambda^{-\frac{1}{2}(n+p)} D^n(\varphi_{\dots}) D^p(\varphi_{\dots}) \mathcal{R}^{\dots} \,,$$

where $\varphi_{\dots}$ denotes some spin-s gauge field, and $\mathcal{R}$ stands for the fluctuation of the Riemann curvature tensor around the AdS background. Such vertices do not admit a $\Lambda \to 0$ limit. The highest order of derivatives which appear in the cubic vertex increases linearly with the spin [12, 13]: $n + p \sim s$. Since all spins $s > 2$ must be included in the nonAbelian HS algebra, the number of derivatives is not bounded from above. In other words, the HS gauge theory is *nonlocal*[1].

The purpose of these lecture notes is to present, in a self-contained[2] way, the nonlinear equations for completely symmetric bosonic HS gauge fields in AdS space of any dimension. The structure of the present lecture notes is as follows.

1.1 Plan of the paper

In Section 2, the MacDowell-Mansouri-Stelle-West formulation of gravity is recalled. In Section 3, some basics about Young tableaux and irreducible tensor representations are introduced. In Section 4, the approach of Section 2 is generalized to HS fields, *i.e.* the free HS gauge theory is formulated as a theory of one-form connections. In Section 5, a nonAbelian HS algebra is constructed. The general definition of a free differential algebra is given in Section 6 and a strategy is explained on how to formulate nonlinear HS field equations in these terms. Section 7 presents the unfolded form of the free massless scalar field and linearized gravity field equations, which are generalized to free HS field equations in Section 8. In Section 9 is explained how the cohomologies of some operator σ_- describe the dynamical content of a theory. The relevant cohomologies are calculated in the HS case in Section 10. Sections 11 and 12 introduce some tools (the star product and the twisted adjoint representation) useful for writing the nonlinear equations, which is done in Section 13. The nonlinear equations are analyzed perturbatively in Section 14 and they are further discussed in Section 15. A brief conclusion inviting to further readings completes these lecture notes. In Appendix 1 some elementary material of lower spin gauge theories is reviewed, while in Appendix 2 some technical points of the nonlinear HS equations are addressed in more details.

For an easier reading of the lecture notes, here is a guide to the regions related to the main topics addressed in these lecture notes:

- Abelian HS gauge theory: In section 4 is reviewed the quadratic actions of free (constrained) HS gauge fields in the metric-like and frame-like approaches.
- Non-Abelian HS algebra: The definition and some properties of the two simplest HS algebras are given in Section 5.
- Unfolded formulation of free HS fields: The unfolding of the free HS equations is very important as a starting point towards nonlinear HS equations at all orders. The general unfolding procedure and its application to the HS gauge theory is explained in many details in Sections 7, 8, 9, 10 and 12. (Sections 9 and 10 can be skipped in a first quick reading since the corresponding material is not necessary for understanding Sections from 11 till 15.)

[1]Nonlocal theories do not automatically suffer from the higher-derivative problem. Indeed, in some cases like string field theory, the problem is somehow cured [21, 19, 22] if the free theory is well-behaved and if nonlocality is treated perturbatively (see [23] for a comprehensive review on this point).

[2]For these lecture notes, the reader is only assumed to have basic knowledge of Yang-Mills theory, general relativity and group theory. The reader is also supposed to be familiar with the notions of differential forms and cohomology groups.

- Non-linear HS equations: Consistent nonlinear equations, that are invariant under the non-Abelian HS gauge transformations and diffeomorphisms, and correctly reproduce the free HS dynamics at the linearized level, are presented and discussed in Sections 6.2, 10.3, 13, 14 and 15.
- Material of wider interest: Sections 2, 3, 6.1, 9 and 11 introduce tools which prove to be very useful in HS gauge theories, but which may also appear in a variety of different contexts.

1.2 Conventions

Our conventions are as follows:

A generic space-time is denoted by $\mathcal{M}^d$ and is a (pseudo)-Riemannian smooth manifold of dimension d, where the metric is taken to be "mostly minus".

Greek letters $\mu, \nu, \rho, \sigma, \ldots$ denote curved (*i.e.,* base) indices, while Latin letters $a, b, c, d, \ldots$ denote fiber indices often referred to as tangent space indices. Both types of indices run from 0 to $d-1$. The tensor η_{ab} is the mostly minus Minkowski metric. Capital Latin letters $A, B, C, D, \ldots$ denote ambient space indices and their range of values is $0, 1, \ldots, d-1, \hat{d}$, where the (timelike) $(d+1)$-th direction is denoted by $\hat{d}$ (in order to distinguish the tangent space index d from the value $\hat{d}$ that it can take). The tensor η_{AB} is diagonal with entries $(+, -, \ldots, -, +)$.

The bracket $[\ldots]$ denotes complete antisymmetrization of indices, with strength one (*e.g.* $A_{[a}B_{b]} = \frac{1}{2}(A_a B_b - A_b B_a)$), while the bracket $\{\ldots\}$ denotes complete symmetrization of the indices, with strength one (*e.g.* $A_{\{a}B_{b\}} = \frac{1}{2}(A_a B_b + A_b B_a)$). Analogously, the commutator and anticommutator are respectively denoted as $[\ ,\ ]$ and $\{\ ,\ \}$.

The de Rham complex $\Omega^*(\mathcal{M}^d)$ is the graded commutative algebra of differential forms that is endowed with the wedge product (the wedge symbol will always be omitted in this paper) and the exterior differential d. $\Omega^p(\mathcal{M}^d)$ is the space of differential p-forms on the manifold $\mathcal{M}^d$, which are sections of the p-th exterior power of the cotangent bundle $T^*\mathcal{M}^d$. In the topologically trivial situation discussed in this paper $\Omega^p(\mathcal{M}^d) = C^\infty(\mathcal{M}^d) \otimes \Lambda^p \mathbb{R}^{d\,*}$, where the space $C^\infty(\mathcal{M}^d)$ is the space of smooth functions from $\mathcal{M}^d$ to $\mathbb{R}$. The generators dx^μ of the exterior algebra $\Lambda\mathbb{R}^{d\,*}$ are Grassmann odd (*i.e.* anticommuting). The exterior differential is defined as $d = dx^\mu \partial_\mu$.

2 Gravity à la MacDowell - Mansouri - Stelle - West

Einstein's theory of gravity is a nonAbelian gauge theory of a spin-2 particle, in a similar way as Yang-Mills theories[3] are nonAbelian gauge theories of spin-1 particles. Local symmetries of Yang-Mills theories originate from internal global symmetries. Similarly, the gauge symmetries of Einstein gravity in the vielbein formulation[4] originate from global space-time symmetries of its most symmetric vacua. The latter symmetries are manifest in the formulation of MacDowell, Mansouri, Stelle and West [25,26].

This section is devoted to the presentation of the latter formulation. In the first subsection 2.1, the Einstein-Cartan formulation of gravity is reviewed and the link with the Einstein-Hilbert action without cosmological constant is explained. A cosmological constant can be introduced into the formalism, which is done in Subsection 2.2. This subsection also contains an elegant action for gravity, written by MacDowell and Mansouri. In Subsection 2.3, the improved version of this action introduced by Stelle and West is presented, the covariance under all symmetries being made manifest.

2.1 Gravity as a Poincaré gauge theory

In this subsection, the frame formulation of gravity with zero cosmological constant is reviewed. We first introduce the dynamical fields and sketch the link to the metric formulation. Then the action is written.

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[3]See Appendix 1.2 for a brief review of Yang-Mills theories.

[4]See *e.g.* [24] for a pedagogical review on the gauge theory formulation of gravity and some of its extensions, like supergravity.

The basic idea is as follows: instead of considering the metric $g_{\mu\nu}$ as the dynamical field, two new dynamical fields are introduced: the vielbein or frame field e^a_μ and the Lorentz connection $\omega^{L\ ab}_\mu$.

The relevant fields appear through the one-forms $e^a = e^a_\mu\, dx^\mu$ and $\omega^{L\, ab} = -\omega^{L\, ba} = \omega^{L\ ab}_\mu\, dx^\mu$. The number of one-forms is equal to $d + \frac{d(d-1)}{2} = \frac{(d+1)d}{2}$, which is the dimension of the Poincaré group $ISO(d-1,1)$. So they can be collected into a single one-form taking values in the Poincaré algebra as $\omega = e^a P_a + \frac{1}{2}\,\omega^{L\, ab} M_{ab}$, where P_a and M_{ab} generate $iso(d-1,1)$ (see Appendix 1.1). The corresponding curvature is the two-form (see Appendix 1.2):

$$R = d\omega + \omega^2 \equiv T^a P_a + \frac{1}{2}\, R^{L\, ab} M_{ab}\,,$$

where T^a is the torsion, given by

$$T^a = D^L e^a = de^a + \omega^{L\ a}{}_b e^b\ ,$$

and $R^{L\ ab}$ is the Lorentz curvature

$$R^{L\ ab} = D^L \omega^{L\ ab} = d\omega^{L\ ab} + \omega^{L\ a}{}_c \omega^{L\ cb}\,,$$

as follows from the Poincaré algebra (1.1)-(1.3).

To make contact with the metric formulation of gravity, one must assume that the frame e^a_μ has maximal rank d so that it gives rise to the nondegenerate metric tensor $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. As will be shown further in this section, one is allowed to require the absence of torsion, $T_a = 0$. Then one solves this constraint and expresses the Lorentz connection in terms of the frame field, $\omega^L = \omega^L(e, \partial e)$. It can be checked that the tensor $R_{\rho\sigma,\,\mu\nu} = e^a_\mu e^b_\nu R^L_{\rho\sigma\ ab}$ is then expressed solely in terms of the metric, and is the Riemann tensor.

The first order action of the frame formulation of gravity is due to Weyl [27]. In any dimension $d > 1$ it can be written in the form

$$S[\, e^a_\mu, \omega^{L\ ab}_\mu] = \frac{1}{2\kappa^2} \int_{\mathcal{M}^d} R^{L\ bc} e^{a_1} \dots e^{a_{d-2}} \epsilon_{a_1 \dots a_{d-2} bc}\ , \tag{2.1}$$

where $\epsilon_{a_1 \dots a_d}$ is the invariant tensor of the special linear group $SL(d)$ and κ^2 is the gravitational constant, so that κ has dimension $(length)^{\frac{d}{2}-1}$. The Euler-Lagrange equations of the Lorentz connection

$$\frac{\delta S}{\delta \omega^{L\ bc}} \propto \epsilon_{a_1 \dots a_{d-2} bc}\, e^{a_1} \dots e^{a_{d-3}} T^{a_{d-2}} = 0$$

imply that the torsion vanishes. The Lorentz connection is then an auxiliary field, which can be removed from the action by solving its own (algebraic) equations of motion. The action $S = S[\, e\,,\, \omega^L(e, \partial e)\,]$ is now expressed only in terms of the vielbein [5]. Actually, only combinations of vielbeins corresponding to the metric appear and the action $S = S[\, g_{\mu\nu}]$ is indeed the second order Einstein-Hilbert action.

The Minkowski space-time solves $R^{L\ ab} = 0$ and $T^a = 0$. It is the most symmetrical solution of the Euler-Lagrange equations, whose global symmetries form the Poincaré group. The gauge symmetries of the action (2.1) are the diffeomorphisms and the local Lorentz transformations. Together, these gauge symmetries correspond to the gauging of the Poincaré group (see Appendix 1.3 for more comments).

2.2 Gravity as a theory of $o(d-1,2)$ gauge fields

In the previous section, the Einstein-Cartan formulation of gravity with vanishing cosmological constant has been presented. We will now show how a nonvanishing cosmological constant can be added to this formalism. In these lectures, we will restrict ourselves to the AdS case but, for the bosonic case we focus on, everything can be rephrased for dS. One is mostly interested in the AdS case for the reason that it

[5] In the context of supergravity, this action principle [18,28] is sometimes called the "1.5 order formalism" [29] because it combines in some sense the virtues of first and second order formalism.

is more suitable for supersymmetric extensions. Furthermore, dS and AdS have rather different unitary representations (for dS there are unitary irreducible representations the energy of which is not bounded from below).

It is rather natural to reinterpret P_a and M_{ab} as the generators of the AdS_d isometry algebra $o(d-1,2)$. The curvature $R = d\omega + \omega^2$ then decomposes as $R = T^a P_a + \frac{1}{2} R^{ab} M_{ab}$, where the Lorentz curvature $R^{L\ ab}$ is deformed to

$$R^{ab} \equiv R^{L\ ab} + R^{cosm\ ab} \equiv R^{L\ ab} + \Lambda\, e^a e^b\,, \tag{2.2}$$

since (1.3) is deformed to (1.4).

MacDowell and Mansouri proposed an action [25], the Lagrangian of which is the (wedge) product of two curvatures (2.2) in $d=4$

$$S^{MM}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^4} R^{a_1 a_2} R^{a_3 a_4}\epsilon_{a_1 a_2 a_3 a_4}\,. \tag{2.3}$$

Expressing R^{ab} in terms of $R^{L\ ab}$ and $R^{cosm\ ab}$ by (2.2), the Lagrangian is the sum of three terms: a term $R^L R^{cosm}$, which is the previous Lagrangian (2.1) without cosmological constant, a cosmological term $R^{cosm}\, R^{cosm}$ and a Gauss-Bonnet term $R^L R^L$. The latter term contains higher-derivatives but it does not contribute to the equations of motion because it is a total derivative.

In any dimension, the AdS_d space-time is defined as the most symmetrical solution of the Euler-Lagrange equations of pure gravity with the cosmological term. As explained in more detail in Section 2.3, it is a solution of the system $R^{ab}=0$, $T^a=0$ such that $\mathrm{rank}(e^a_\mu)=d$.

The MacDowell-Mansouri action admits a higher dimensional generalization [30]

$$S^{MM}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^d} R^{a_1 a_2} R^{a_3 a_4} e^{a_5}\ldots e^{a_d}\epsilon_{a_1\ldots a_d}\,. \tag{2.4}$$

Because the Gauss-Bonnet term

$$S^{GB}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^d} R^{L\,a_1 a_2} R^{L\,a_3 a_4} e^{a_5}\ldots e^{a_d}\epsilon_{a_1\ldots a_d}$$

is not topological beyond $d=4$, the field equations resulting from the action (2.4) are different from the Einstein equations in d dimensions. However the difference is by nonlinear terms that do not contribute to the free spin-2 equations [30] apart from replacing the cosmological constant Λ by $\frac{2(d-2)}{d}\Lambda$ (in such a way that no correction appears in $d=4$, as expected). One way to see this is by considering the action

$$S^{nonlin}[\,e,\omega] \equiv S^{GB}[\,e,\omega] + \frac{d-4}{4\kappa^2}\int_{\mathcal{M}^d}\Big(\frac{2}{d-2}\, R^{L\,a_1 a_2} e^{a_3}\ldots e^{a_d} + \frac{\Lambda}{d}\, e^{a_1}\ldots e^{a_d}\Big)\epsilon_{a_1\ldots a_d}\,, \tag{2.5}$$

which is the sum of the Gauss-Bonnet term plus terms of the same type as the Einstein-Hilbert and cosmological terms (note that the latter are absent when $d=4$). The variation of (2.5) is equal to

$$\delta S^{nonlin}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^d} R^{a_1 a_2} R^{a_3 a_4}\delta(e^{a_5}\ldots e^{a_d})\,\epsilon_{a_1\ldots a_d}\,, \tag{2.6}$$

when the torsion is required to be zero (*i.e.* applying the 1.5 order formalism to see that the variation over the Lorentz connection does not contribute). Indeed, the variation of the action (2.5) vanishes when $d=4$, but when $d>4$ the variation (2.6) is bilinear in the AdS_d field strength R^{ab}. Since the AdS_d field strength is zero in the vacuum AdS_d solution, variation of the action S^{nonlin} is nonlinear in the fluctuations near the AdS_d background. As a consequence, at the linearized level the Gauss-Bonnet term does not affect the form of the free spin-2 equations of motion, it merely redefines an overall factor in front of the action and the cosmological constant via $\kappa^2 \to (\frac{d}{2}-1)\kappa^2$ and $\Lambda \to \frac{2(d-2)}{d}\Lambda$, respectively (as can be seen by substituting S^{GB} in (2.4) with its expression in terms of S^{nonlin} from (2.5)). Also let us note that from (2.6) it is obvious that the variation of the Gauss-Bonnet term contains second order

derivatives of the metric, *i.e.* it is of the Lanczos-Lovelock type (see [24] for a review).

Beyond the free field approximation the corrections to Einstein's field equations resulting from the action (2.4) are nontrivial for $d > 4$ and nonanalytic in Λ (as can be seen from (2.6)), having no meaningful flat limit. As will be shown later, this is analogous to the structure of HS interactions which also contain terms with higher derivatives and negative powers of Λ. The important difference is that in the case of gravity one can subtract the term (2.5) without destroying the symmetries of the model, while this is not possible in the HS gauge theories. In both cases, the flat limit $\Lambda \to 0$ is perfectly smooth at the level of the algebra (*e.g.* $o(d-1,2) \to iso(d-1,1)$ for gravity, see Appendix 1.1) and at the level of the free equations of motion, but it may be singular at the level of the action and nonlinear field equations.

2.3 MacDowell-Mansouri-Stelle-West gravity

The action (2.4) is not manifestly $o(d-1,2)$ gauge invariant. Its gauge symmetries are the diffeomorphisms and the local Lorentz transformations. It is possible to make the $o(d-1,2)$ gauge symmetry manifest by combining the vielbein and the Lorentz connection into a single field $\omega = dx^\mu \omega_\mu{}^{AB} M_{AB}$ and by introducing a vector V^A called *compensator*[6]. The fiber indices A, B now run from 0 to d. They are raised and lowered by the invariant mostly minus metric η_{AB} of $o(d-1,2)$ (see Appendix 1.1).

In this subsection, the MacDowell-Mansouri-Stelle-West (MMSW) action [26] is written and it is shown how to recover the action presented in the previous subsection. The particular vacuum solution which corresponds to AdS space-time is also introduced. Finally the symmetries of the MMSW action and of the vacuum solution are analyzed.

To have a formulation with manifest $o(d-1,2)$ gauge symmetries, a time-like vector compensator V^A has to be introduced which is constrained to have a constant norm ρ,

$$V^A V^B \eta_{AB} = \rho^2 \,. \tag{2.7}$$

As one will see, the constant ρ is related to the cosmological constant by

$$\rho^2 = -\Lambda^{-1} \,. \tag{2.8}$$

The MMSW action is ([26] for $d = 4$ and [30] for arbitrary d)

$$S^{MMSW}[\,\omega^{AB}, V^A] = -\frac{\rho}{4\kappa^2} \int_{\mathcal{M}^d} \epsilon_{A_1 \dots A_{d+1}} R^{A_1 A_2} R^{A_3 A_4} E^{A_5} \dots E^{A_d} V^{A_{d+1}} \,, \tag{2.9}$$

where the curvature or field strength R^{AB} is defined by

$$R^{AB} \equiv d\omega^{AB} + \omega^{AC} \omega_C{}^B$$

and the frame field E^A by

$$E^A \equiv DV^A = dV^A + \omega^A_{\ B} V^B \,.$$

Furthermore, in order to make link with Einstein gravity, two constraints are imposed: (i) the norm of V^A is fixed, and (ii) the frame field E^A_μ is assumed to have maximal rank equal to d. As the norm of V^A is constant, the frame field satisfies

$$E^A V_A = 0 \,. \tag{2.10}$$

If the condition (2.7) is relaxed, then the norm of V^A corresponds to an additional dilaton-like field [26].

Let us now analyze the symmetries of the MMSW action. The action is manifestly invariant under

[6] This compensator field compensates additional symmetries serving for them as a Higgs field. The terminology is borrowed from application of conformal supersymmetry for the analysis of Poincaré supermultiplets (see *e.g.* [31]). It should *not* be confused with the homonymous - but unrelated - gauge field introduced in another approach to free HS fields [32, 33].

- Local $o(d-1,2)$ transformations:

$$\delta\omega_\nu^{AB}(x) = D_\nu\epsilon^{AB}(x)\,,\qquad \delta V^A(x) = -\epsilon^{AB}(x)V_B(x)\,;\tag{2.11}$$

- Diffeomorphisms:

$$\delta\omega_\nu^{AB}(x) = \partial_\nu\xi^\mu(x)\,\omega_\mu^{AB}(x)\,+\,\xi^\mu(x)\,\partial_\mu\omega_\nu^{AB}(x)\,,\qquad \delta V^A(x) = \xi^\nu(x)\,\partial_\nu V^A(x)\,.\tag{2.12}$$

Let us define the covariantized diffeomorphism as the sum of a diffeomorphism with parameter ξ^μ and a $o(d-1,2)$ local transformation with parameter $\epsilon^{AB}(\xi^\mu) = -\xi^\mu\omega_\mu^{AB}$. The action of this transformation is thus

$$\delta^{cov}\omega_\mu^{AB} = \xi^\nu R_{\nu\mu}^{AB}\,,\qquad \delta^{cov}V^A = \xi^\nu E_\nu^A\tag{2.13}$$

by (2.11)-(2.12).

The compensator vector is pure gauge. Indeed, by local $O(d-1,2)$ rotations one can gauge fix $V^A(x)$ to any value with $V^A(x)V_A(x) = \rho^2$. In particular, one can reach the standard gauge

$$V^A = \rho\,\delta^A_{\hat d}\,.\tag{2.14}$$

Taking into account (2.10), one observes that the covariantized diffeomorphism also makes it possible to gauge fix fluctuations of the compensator $V^A(x)$ near any fixed value. Because the full list of symmetries can be represented as a combination of covariantized diffeomorphisms, local Lorentz transformations and diffeomorphisms, in the standard gauge (2.14) the gauge symmetries are spontaneously broken to the $o(d-1,1)$ local Lorentz symmetry and diffeomorphisms. In the standard gauge, one therefore recovers the field content and the gauge symmetries of the MacDowell-Mansouri action. Let us note that covariantized diffeomorphisms (2.13) do not affect the connection ω_μ^{AB} if it is flat (*i.e.* has zero curvature $R_{\nu\mu}^{AB}$). In particular covariantized diffeomorphisms do not affect the background AdS geometry.

To show the equivalence of the action (2.9) with the action (2.4), it is useful to define a Lorentz connection by

$$\omega^{L\;AB} \equiv \omega^{AB} - \rho^{-2}(E^AV^B - E^BV^A)\,.\tag{2.15}$$

In the standard gauge, the curvature can be expressed in terms of the vielbein $e^a \equiv E^a = \rho\,\omega^{a\hat d}$ and the nonvanishing components of the Lorentz connection $\omega^{L\;ab} = \omega^{ab}$ as

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^{aC}\omega_C{}^b = d\omega^{L\;ab} + \omega^{L\;a}{}_c\omega^{L\;cb} - \rho^{-2}e^ae^b = R^{L\;ab} + R^{cosm\;ab}\,,\\ R^{a\hat d} &= d\omega^{a\hat d} + \omega^{ac}\omega_c{}^{\hat d} = \rho^{-1}T^a\,. \end{aligned}$$

Inserting these gauge fixed expressions into the MMSW action yields the action (2.4), where $\Lambda = -\rho^{-2}$. The MMSW action is thus equivalent to (2.4) by partially fixing the gauge invariance. Let us note that a version of the covariant compensator formalism applicable to the case with zero cosmological constant was developed in [34].

Let us now consider the vacuum equations $R^{AB}(\omega_0) = 0$. They are equivalent to $T^a(\omega_0) = 0$ and $R^{ab}(\omega_0) = 0$ and, under the condition that $\mathrm{rank}(E_\nu^A) = d$, they uniquely define the local geometry of AdS_d with parameter ρ, in a coordinate independent way. The solution ω_0 also obviously satisfies the equations of motion of the MMSW action. To find the symmetries of the vacuum solution ω_0, one first notes that vacuum solutions are sent onto vacuum solutions by diffeomorphisms and local AdS transformations, because they transform the curvature homogeneously. Since covariantized diffeomorphisms do not affect ω_0, to find symmetries of the chosen solution ω_0 it is enough to check its transformation law under local $o(d-1,2)$ transformation. Indeed, by adjusting an appropriate covariantized diffeomorphism it is always possible to keep the compensator invariant.

The solution ω_0 is invariant under those $o(d-1,2)$ gauge transformations for which the parameter $\epsilon^{AB}(x)$ satisfies

$$0 = D_0\epsilon^{AB}(x) = d\epsilon^{AB}(x) + \omega_0{}^A{}_C(x)\,\epsilon^{CB}(x) - \omega_0{}^B{}_C(x)\,\epsilon^{CA}(x)\,.\tag{2.16}$$

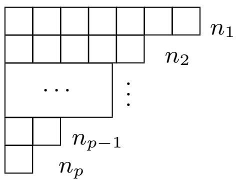
This equation fixes the derivatives $\partial_\mu \epsilon^{AB}(x)$ in terms of $\epsilon^{AB}(x)$ itself. In other words, once $\epsilon^{AB}(x_0)$ is chosen for some x_0, $\epsilon^{AB}(x)$ can be reconstructed for all x in a neighborhood of x_0, since by consistency[7] *all* derivatives of the parameter can be expressed as functions of the parameter itself. The parameters $\epsilon^{AB}(x_0)$ remain arbitrary, being parameters of the global symmetry $o(d-1,2)$. This means that, as expected for AdS_d space-time, the symmetry of the vacuum solution ω_0 is the global $o(d-1,2)$.

The lesson is that, to describe a gauge model that has a global symmetry h, it is useful to reformulate it in terms of the gauge connections ω and curvatures R of h in such a way that the zero curvature condition $R=0$ solves the field equations and provides a solution with h as its global symmetry. If a symmetry h is not known, this observation can be used the other way around: by reformulating the dynamics à la MacDowell-Mansouri one might guess the structure of an appropriate curvature R and thereby the nonAbelian algebra h.

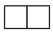
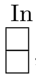
3 Young tableaux and Howe duality

In this section, the Young tableaux are introduced. They characterize the irreducible representations of $gl(M)$ and $o(M)$. A representation of these algebras that will be useful in the sequel is built.

A *Young tableau* $\{n_i\}$ $(i=1,\dots,p)$ is a diagram which consists of a finite number $p>0$ of rows of identical squares. The lengths of the rows are finite and do not increase: $n_1 \geqslant n_2 \geqslant \dots \geqslant n_p \geqslant 0$. The Young tableau $\{n_i\}$ is represented as follows:



Let us consider covariant tensors of $gl(M)$: $A_{a,\,b,\,c,\dots}$ where $a,b,c,\dots = 1,2,\dots,M$. Simple examples of these are the symmetric tensor $A^S_{a,\,b}$ such that $A^S_{a,\,b} - A^S_{b,\,a} = 0$, or the antisymmetric tensor $A^A_{a,\,b}$ such that $A^A_{a,\,b} + A^A_{b,\,a} = 0$.

A complete set of covariant tensors irreducible under $gl(M)$ is given by the tensors $A_{a^1_1 \dots a^1_{n_1},\,\dots\,,\,a^p_1 \dots a^p_{n_p}}$ $(n_i \geq n_{i+1})$ that are symmetric in each set of indices $\{a^i_1 \dots a^i_{n_i}\}$ with fixed i and that vanish when one symmetrizes the indices of a set $\{a^i_1 \dots a^i_{n_i}\}$ with any index a^j_l with $j>i$. The properties of these irreducible tensors can be conveniently encoded into Young tableaux. The Young tableau $\{n_i\}$ $(i=1,\dots,p)$ is associated with the tensor $A_{a^1_1 \dots a^1_{n_1},\,\dots\,,\,a^p_1 \dots a^p_{n_p}}$. Each box of the Young tableau is related to an index of the tensor, boxes of the same row corresponding to symmetric indices. Finally, the symmetrization of all the indices of a row with an index from any row below vanishes. In this way, the irreducible tensors A^S_{ab} and $A^A_{a,\,b}$ are associated with the Young tableaux  and , respectively.

Let us introduce the polynomial algebra $\mathbb{R}[Y^a_i]$ generated by commuting generators Y^a_i where $i = 1,\dots p$; $a = 1,\dots M$. Elements of the algebra $\mathbb{R}[Y^a_i]$ are of the form

$$A(Y) = A_{a^1_1 \dots a^1_{n_1},\,\dots\,,\,a^p_1 \dots a^p_{n_p}} Y_1^{a^1_1} \dots Y_1^{a^1_{n_1}} \dots Y_p^{a^p_1} \dots Y_p^{a^p_{n_p}} .$$

[7]The identity $D_0^2 = R_0 = 0$ ensures consistency of the system (2.16), which is overdetermined because it contains $\frac{d^2(d+1)}{2}$ equations for $\frac{d(d+1)}{2}$ unknowns. Consistency in turn implies that higher space-time derivatives $\partial_{\nu_1} \dots \partial_{\nu_n} \epsilon^{AB}(x)$ obtained by hitting (2.16) $n-1$ times with $D_{0\nu_k}$ are guaranteed to be symmetric in the indices $\nu_1 \dots \nu_n$.

The condition that A is irreducible under $gl(M)$, *i.e.* is a Young tableau $\{n_i\}$, can be expressed as

$$\begin{aligned} Y_i^a \frac{\partial}{\partial Y_i^a} A(Y) &= n_i A(Y)\,, \\ Y_i^b \frac{\partial}{\partial Y_j^b} A(Y) &= 0\,, \qquad i<j\,, \end{aligned} \tag{3.1}$$

where no sum on i is to be understood in the first equation. Let us first note that, as the generators Y_i^a commute, the tensor A is automatically symmetric in each set of indices $\{a_1^i \dots a_{n_i}^i\}$. The operator on the l.h.s. of the first equation of (3.1) then counts the number of Y_i^a's, the index i being fixed and the index a arbitrary. The first equation thus ensures that there are n_i indices contracted with Y_i, forming the set $a_1^i \dots a_{n_i}^i$. The second equation of (3.1) is equivalent to the vanishing of the symmetrization of a set of indices $\{a_1^i \dots a_{n_i}^i\}$ with an index b to the right. Indeed, the operator on the l.h.s. replaces a generator Y_j^b by a generator Y_i^b, $j>i$, thus projecting $A_{\dots,\, a_1^i \dots a_{n_i}^i, \dots b \dots}$ on its component symmetric in $\{a_1^i \dots a_{n_i}^i b\}$.

Two types of generators appear in the above equations: the generators

$$t^a_b = Y_i^a \frac{\partial}{\partial Y_i^b} \tag{3.2}$$

of $gl(M)$ and the generators

$$l_i^j = Y_i^a \frac{\partial}{\partial Y_j^a} \tag{3.3}$$

of $gl(p)$. These generators commute

$$[l_i^j, t^a_b] = 0 \tag{3.4}$$

and the algebras $gl(p)$ and $gl(M)$ are said to be *Howe dual* [35]. The important fact is that the irreducibility conditions (3.1) of $gl(M)$ are the highest weight conditions with respect to $gl(p)$.

When all the lengths n_i have the same value n, $A(Y)$ is invariant under $sl(p) \subset gl(p)$. Moreover, exchange of any two rows of this rectangular Young tableau only brings a sign factor $(-1)^n$, as is easy to prove combinatorically. The conditions (3.1) are then equivalent to:

$$\left(Y_i^a \frac{\partial}{\partial Y_j^a} - \frac{1}{p}\delta_i^j Y_k^a \frac{\partial}{\partial Y_k^a} \right) A(Y) = 0\;. \tag{3.5}$$

Indeed, let us first consider the equation for $i=j$. The operator $Y_k^a \frac{\partial}{\partial Y_k^a}$ (where there is a sum over k and a) counts the total number m of Y's in $A(Y)$, while $Y_i^a \frac{\partial}{\partial Y_i^a}$ counts the number of Y_i's for some fixed i. The condition can only be satisfied if $\frac{m}{p}$ is an integer, *i.e.* $m=np$ for some integer n. As the condition is true for all i's, there are thus n Y_i's for every i. In other words, the tensorial coefficient of $A(Y)$ has p sets of n indices, *i.e.* is rectangular. The fact that it is a Young tableau is insured by the condition (3.5) for $i<j$, which is simply the second condition of (3.1). That the condition (3.5) is true both for $i<j$ and for $i>j$ is a consequence of the simple fact that any finite-dimensional $sl(p)$ module with zero $sl(p)$ weights (which are differences of lengths of the rows of the Young tableau) is $sl(p)$ invariant. Alternatively, this follows from the property that exchange of rows leaves a rectangular Young tableau invariant.

If there are p_1 rows of length n_1, p_2 rows of length n_2, etc. , then $A(Y)$ is invariant under $sl(p_1) \oplus sl(p_2) \oplus sl(p_3) \oplus \dots$, as well as under permutations within each set of p_i rows of length n_i.

To construct irreducible representations of $o(M-N,N)$, one needs to add the condition that A is

traceless[8], which can be expressed as:

$$\frac{\partial^2}{\partial Y_i^a \partial Y_j^b}\, \eta^{ab}\, A(Y) = 0\,, \qquad \forall\, i,j\,, \tag{3.6}$$

where η_{ab} is the invariant metric of $o(M-N,N)$. The generators of $o(M-N,N)$ are given by

$$t_{ab} = \frac{1}{2}(\eta_{ac} t^c_b - \eta_{bc} t^c_a)\,. \tag{3.7}$$

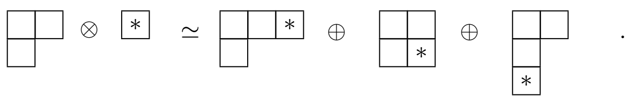
They commute with the generators

$$k_{ij} = \eta_{ab} Y_i^a Y_j^b\,, \quad l_i^j = Y_i^a \frac{\partial}{\partial Y_j^a}\,, \quad m^{ij} = \eta^{ab} \frac{\partial^2}{\partial Y_i^a \partial Y_j^b} \tag{3.8}$$

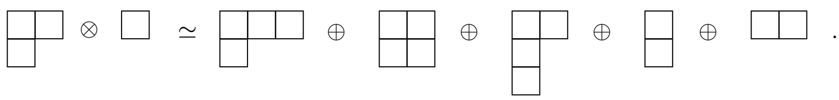
of $sp(2p)$. The conditions (3.1) and (3.6) are highest weight conditions for the algebra $sp(2p)$ which is Howe dual to $o(M-N,N)$.

In the notation developed here, the irreducible tensors are manifestly symmetric in groups of indices. This is a convention: one could as well choose to have manifestly antisymmetric groups of indices corresponding to columns of the Young tableau. An equivalent implementation of the conditions for a tensor to be a Young tableau can be performed in the antisymmetric convention, by taking fermionic generators Y.

To end up with this introduction to Young diagrams, we give two "multiplication rules" of one box with an arbitrary Young tableau. More precisely, the tensor product of a vector (characterized by one box) with an irreducible tensor under $gl(M)$ characterized by a given Young tableau decomposes as the direct sum of irreducible tensors under $gl(M)$ corresponding to all possible Young tableaux obtained by adding one box to the initial Young tableau, e.g.



For the (pseudo)orthogonal algebras $o(M-N,N)$, the tensor product of a vector (characterized by one box) with a traceless tensor characterized by a given Young tableau decomposes as the direct sum of traceless tensors under $o(M-N,N)$ corresponding to all possible Young tableaux obtained by adding or removing one box from the initial Young tableau (a box can be removed as a result of contraction of indices), e.g.



4 Free symmetric higher spin gauge fields as one-forms

Properties of HS gauge theories are to a large extent determined by the HS global symmetries of their most symmetric vacua. The HS symmetry restricts interactions and fixes spectra of spins of massless fields in HS theories as ordinary supersymmetry does in supergravity. To elucidate the structure of a global HS algebra h it is useful to follow the approach in which fields, action and transformation laws are formulated in terms of the connection of h.

[8]For $M = 2N$ modulo 4, the irreducibility conditions also include the (anti)selfduality conditions on the tensors described by Young tableaux with $M/2$ rows. However, these conditions are not used in the analysis of HS dynamics in this paper.

4.1 Metric-like formulation of higher spins

The free HS gauge theories were originally formulated in terms of completely symmetric and double-traceless HS-fields $\varphi_{\nu_1\dots\nu_s}$ [36, 37], in a way analogous to the metric formulation of gravity (see [38] for recent reviews on the metric-like formulation of HS theories). In Minkowski space-time $\mathbb{R}^{d-1,1}$, the spin-s Fronsdal action is

$$\begin{aligned} S_2^{(s)}[\varphi] \;=\; & \frac{1}{2}\int d^dx\,\Big(\partial_\nu\varphi_{\mu_1\dots\mu_s}\partial^\nu\varphi^{\mu_1\dots\mu_s} \\ & -\frac{s(s-1)}{2}\,\partial_\nu\varphi^\lambda{}_{\lambda\mu_3\dots\mu_s}\partial^\nu\varphi_\rho{}^{\rho\mu_3\dots\mu_s}+s(s-1)\,\partial_\nu\varphi^\lambda{}_{\lambda\mu_3\dots\mu_s}\partial_\rho\varphi^{\nu\rho\mu_3\dots\mu_s} \\ & -s\,\partial_\nu\varphi^\nu{}_{\mu_2\dots\mu_s}\partial_\rho\varphi^{\rho\mu_2\dots\mu_s}-\frac{s(s-1)(s-2)}{4}\,\partial_\nu\varphi^{\nu\rho}{}_{\rho\mu_2\dots\mu_s}\partial_\lambda\varphi_\sigma{}^{\lambda\sigma\mu_2\dots\mu_s}\Big)\,, \end{aligned} \tag{4.1}$$

where the metric-like field is double-traceless ($\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\varphi_{\mu_1\dots\mu_s}=0$) and has the dimension of $(length)^{1-d/2}$. This action is invariant under Abelian HS gauge transformations

$$\delta\varphi_{\mu_1\dots\mu_s}=\partial_{\{\mu_1}\epsilon_{\mu_2\dots\mu_s\}}\;, \tag{4.2}$$

where the gauge parameter is a completely symmetric and traceless rank-$(s-1)$ tensor,

$$\eta^{\mu_1\mu_2}\epsilon_{\mu_1\dots\mu_{s-1}}=0\;.$$

For spin $s=2$, (4.1) is the Pauli-Fierz action that is obtained from the linearization of the Einstein-Hilbert action via $g_{\mu\nu}=\eta_{\mu\nu}+\kappa\varphi_{\mu\nu}$ and the gauge transformations (4.2) correspond to linearized diffeomorphisms. The approach followed in this section is to generalize the MMSW construction of gravity to the case of free HS gauge fields in AdS backgrounds. The free HS dynamics will then be expressed in terms of one-form connections taking values in certain representations of the AdS_d isometry algebra $o(d-1,2)$.

The procedure is similar to that of Section 2 for gravity. The HS metric-like field is replaced by a frame-like field and a set of connections. These new fields are then united in a single connection. The action is given in terms of the latter connection and a compensator vector. It is constructed in such a way that it reproduces the dynamics of the Fronsdal formulation, once auxiliary fields are removed and part of the gauge invariance is fixed.

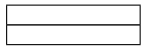
4.2 Frame-like formulation of higher spins

The double-traceless metric-like HS gauge field $\varphi_{\mu_1\dots\mu_s}$ is replaced by a frame-like field $e_\mu{}^{a_1\dots a_{s-1}}$, a Lorentz-like connection $\omega_\mu{}^{a_1\dots a_{s-1},\,b}$ [39] and a set of connections $\omega_\mu{}^{a_1\dots a_{s-1},\,b_1\dots b_t}$ called *extra fields*, where $t=2,\dots,s-1$, $s>2$ [40, 41]. All fields e, ω are traceless in the fiber indices a,b, which have the symmetry of the Young tableaux $s-1$ t, where $t=0$ for the frame-like field and $t=1$ for the Lorentz-like connection. The metric-like field arises as the completely symmetric part of the frame field [39],

$$\varphi_{\mu_1\dots\mu_s}=\rho^{3-s-\frac{d}{2}}e_{\{\mu_1,\,\mu_2\dots\mu_s\}}\,,$$

where the dimensionful factor of $\rho^{3-s-\frac{d}{2}}$ is introduced for the future convenience (ρ is a length scale) and all fiber indices have been lowered using the AdS or flat frame field $e_0{}^a_\mu$ defined in Section 2. From the fiber index tracelessness of the frame field follows automatically that the field $\varphi_{\mu_1\dots\mu_s}$ is double traceless.

The frame-like field and other connections are then combined [30] into a connection one-form $\omega^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ (where $A,B=0,\dots,d-1,\hat{d}$) taking values in the irreducible $o(d-1,2)$-module

characterized by the two-row traceless rectangular Young tableau  of length $s-1$, that is

$$\omega_\mu^{A_1\dots A_{s-1},B_1\dots B_{s-1}} = \omega_\mu^{\{A_1\dots A_{s-1}\},B_1\dots B_{s-1}} = \omega_\mu^{A_1\dots A_{s-1},\{B_1\dots B_{s-1}\}}\,,$$
$$\omega_\mu^{\{A_1\dots A_{s-1},A_s\}\,B_2\dots B_{s-1}} = 0\,,\quad \omega_\mu{}^{A_1\dots A_{s-3}\,C}{}_{C,}{}^{B_1\dots B_{s-1}} = 0\,. \tag{4.3}$$

One also introduces a time-like vector V^A of constant norm ρ. The component of the connection $\omega^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ that is most parallel to V^A is the frame-like field

$$E^{A_1\dots A_{s-1}} = \omega^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}} V_{B_1}\dots V_{B_{s-1}}\,,$$

while the less V-longitudinal components are the other connections. Note that the contraction of the connection with more than $s-1$ compensators V^A is zero by virtue of (4.3). Let us be more explicit in a specific gauge. As in the MMSW gravity reformulation, one can show that V^A is a pure gauge field and that one can reach the standard gauge $V^A = \delta^A_{\hat d}\,\rho$ (the argument will not be repeated here). In the standard gauge, the frame field and the connections are given by

$$\begin{aligned} e^{\,a_1\dots a_{s-1}} &= \rho^{s-1}\omega^{\,a_1\dots a_{s-1},\,\hat d\dots\hat d}\,,\\ \omega^{\,a_1\dots a_{s-1},\,b_1\dots b_t} &= \rho^{s-1-t}\Pi(\omega^{\,a_1\dots a_{s-1},\,b_1\dots b_t\,\hat d\dots\hat d})\,, \end{aligned}$$

where the powers of ρ originate from a corresponding number of contractions with the compensator vector V^A and Π is a projector to the Lorentz-traceless part of a Lorentz tensor, which is needed for $t\geq 2$. These normalization factors are consistent with the fact that the auxiliary fields $\omega_\mu{}^{a_1\dots a_{s-1},\,b_1\dots b_t}$ will be found to be expressed via t partial derivatives of the frame field $e_\mu{}^{a_1\dots a_{s-1}}$ at the linearized level.

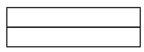
The linearized field strength or curvature is defined as the $o(d-1,2)$ covariant derivative of the connection $\omega^{A_1\dots A_{s-1},B_1\dots B_{s-1}}$, *i.e.* by

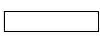
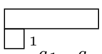
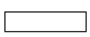
$$\begin{aligned} R_1^{A_1\dots A_{s-1},B_1\dots B_{s-1}} &= D_0\omega^{A_1\dots A_{s-1},B_1\dots B_{s-1}}\\ &= d\omega^{A_1\dots A_{s-1},B_1\dots B_{s-1}} + \omega_0{}^{A_1}{}_C\,\omega^{CA_2\dots A_{s-1},\,B_1\dots B_{s-1}} + \dots\\ &\qquad + \omega_0{}^{B_1}{}_C\,\omega^{A_1\dots A_{s-1},\,CB_2\dots B_{s-1}} + \dots\,, \end{aligned} \tag{4.4}$$

where the dots stand for the terms needed to get an expression symmetric in $A_1\dots A_{s-1}$ and $B_1\dots B_{s-1}$, and $\omega_0{}^A{}_B$ is the $o(d-1,2)$ connection associated to the AdS space-time solution, as defined in Section 2. The connection $\omega_\mu^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ has dimension $(length)^{-1}$ in such a way that the field strength $R_{\mu\nu}^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ has proper dimension $(length)^{-2}$.

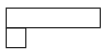
As $(D_0)^2 = R_0 = 0$, the linearized curvature R_1 is invariant under Abelian gauge transformations of the form

$$\delta_0\omega^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}} = D_0\epsilon^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}\,. \tag{4.5}$$

The gauge parameter $\epsilon^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ has the symmetry  and is traceless.

Before writing the action, let us analyze the frame field and its gauge transformations, in the standard gauge. According to the multiplication rule formulated in the end of Section 3, the frame field $e_\mu{}^{a_1\dots a_{s-1}}$ contains three irreducible (traceless) Lorentz components characterized by the symmetry of their indices:  $_s$,  $^{s-1}_{1}$ and  $_{s-2}$, where the last tableau describes the trace component of the frame field $e_\mu{}^{a_1\dots a_{s-1}}$. Its gauge transformations are given by (4.5) and read

$$\delta_0 e^{a_1\dots a_{s-1}} = D_0^L\epsilon^{a_1\dots a_{s-1}} - e_{0\ c}\,\epsilon^{a_1\dots a_{s-1},\,c}\,.$$

The parameter $\epsilon^{a_1\dots a_{s-1},\,c}$ is a generalized local Lorentz parameter. It allows us to gauge away the traceless component  of the frame field. The other two components of the latter just correspond

to a completely symmetric double traceless Fronsdal field $\varphi_{\mu_1\dots\mu_s}$. The remaining invariance is then the Fronsdal gauge invariance (4.2) with a traceless completely symmetric parameter $\epsilon^{a_1\dots a_{s-1}}$.

4.3 Action of higher spin gauge fields

For a given spin s, the most general $o(d-1,2)$-invariant action that is quadratic in the linearized curvatures (4.4) and, for the rest, built only from the compensator V^C and the background frame field $E_0^B = D_0 V^B$ is

$$S_2^{(s)}[\,\omega_\mu^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}\,,\omega_0^{AB}\,,V^C\,] \;=\; \frac{1}{2}\sum_{p=0}^{s-2} a(s,p)\,S^{(s,p)}[\,\omega_\mu^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}\,,\omega_0^{AB}\,,\,V^C\,]\,, \qquad (4.6)$$

where $a(s,p)$ is the *a priori* arbitrary coefficient of the term

$$\begin{aligned} S^{(s,p)}[\,\omega,\omega_0,V\,] \quad = \quad & \epsilon_{A_1\dots A_{d+1}}\int_{M^d} E_0^{A_5}\dots E_0^{A_d}V^{A_{d+1}}V_{C_1}\dots V_{C_{2(s-2-p)}}\,\times \\ & \times R_1^{A_1B_1\dots B_{s-2},\,A_2C_1\dots C_{s-2-p}D_1\dots D_p}\,R_1{}^{A_3}{}_{B_1\dots B_{s-2},}{}^{A_4C_{s-1-p}\dots C_{2(s-2-p)}}{}_{D_1\dots D_p}\,. \end{aligned}$$

This action is manifestly invariant under diffeomorphisms, local $o(d-1,2)$ transformations (2.11) and Abelian HS gauge transformations (4.5), which leave invariant the linearized HS curvatures (4.4). Having fixed the AdS_d background gravitational field ω_0^{AB} and compensator V^A, the diffeomorphisms and the local $o(d-1,2)$ transformations break down to the AdS_d global symmetry $o(d-1,2)$.

As will be explained in Sections 6 and 8, the connections $\omega_\mu{}^{a_1\dots a_{s-1},\,b_1\dots b_t}$ can be expressed as t derivatives of the frame-like field, via analogues of the torsion constraint. Therefore, to make sure that higher-derivative terms are absent from the free theory, the coefficients $a(s,p)$ are chosen in such a way that the Euler-Lagrange derivatives are nonvanishing only for the frame field and the first connection ($t=1$). All extra fields, *i.e.* the connections $\omega_\mu{}^{a_1\dots a_{s-1},\,b_1\dots b_t}$ with $t>1$, should appear only through total derivatives[9]. This requirement fixes uniquely the spin-s free action up to a coefficient $b(s)$ in front of the action. More precisely, the coefficient $a(s,p)$ is essentially a relative coefficient given by [30]

$$a(s,p) = b(s)(-\Lambda)^{-(s-p-1)}\,\frac{(d-5+2(s-p-2))!!\,(s-p-1)}{(s-p-2)!}\,,$$

where $b(s)$ is the arbitrary spin-dependent factor.

The equations of motion for $\omega_\mu{}^{a_1\dots a_{s-1},\,b}$ are equivalent to the "zero-torsion condition"

$$R_{1\,A_1\dots A_{s-1},\,B_1\dots B_{s-1}}V^{B_1}\dots V^{B_{s-1}} = 0\,.$$

They imply that $\omega_\mu{}^{a_1\dots a_{s-1},\,b}$ is an auxiliary field that can be expressed in terms of the first derivative of the frame field modulo a pure gauge part associated with the symmetry parameter $\epsilon_\mu{}^{a_1\dots a_{s-1},\,b_1b_2}$. Substituting the found expression for $\omega_\mu{}^{a_1\dots a_{s-1},\,b}$ into the HS action yields an action only expressed in terms of the frame field and its first derivative, modulo total derivatives. As gauge symmetries told us, the action actually depends only on the completely symmetric part of the frame field, *i.e.* the Fronsdal field. Moreover, the action (4.6) has the same gauge invariance as Fronsdal's one, thus it must be proportional to the Fronsdal action (4.1) because the latter is fixed up to a front factor by the requirements of being gauge invariant and of being of second order in the derivatives of the field [42].

[9]The extra fields show up in the nonlinear theory and are responsible for the higher-derivatives as well as for the terms with negative powers of Λ in the interaction vertices.

5 Simplest higher spin algebras

In the previous section, the dynamics of free HS gauge fields has been expressed as a theory of one-forms, the $o(d-1,2)$ fiber indices of which have symmetries characterized by two-row rectangular Young tableaux. This suggests that there exists a nonAbelian HS algebra $h \supset o(d-1,2)$ that admits a basis formed by a set of elements $T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$ in irreducible representations of $o(d-1,2)$ characterized by such Young tableaux. More precisely, the basis elements $T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$ satisfy the following properties $T_{\{A_1\ldots A_{s-1},A_s\}B_2\ldots B_{s-1}} = 0$, $T_{A_1\ldots A_{s-3}CC,}{}^{B_1\ldots B_{s-1}} = 0$, and the basis contains the $o(d-1,2)$ basis elements $T_{A,B} = -T_{B,A}$ such that all generators transform as $o(d-1,2)$ tensors

$$[T_{C,D}\,,\, T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}] = \eta_{DA_1} T_{CA_2\ldots A_{s-1},B_1\ldots B_{s-1}} + \ldots\,. \tag{5.1}$$

The question is whether a nonAbelian algebra h with these properties really exists. If yes, the Abelian curvatures R_1 can be understood as resulting from the linearization of the nonAbelian field curvatures $R = dW + W^2$ of h with the h gauge connection $W = \omega_0 + \omega$, where ω_0 is some fixed flat (*i.e.* vanishing curvature) zero-order connection of the subalgebra $o(d-1,2) \subset h$ and ω is the first-order dynamical part which describes massless fields of various spins.

The HS algebras with these properties were originally found for the case of AdS_4 [43, 44, 45, 46] in terms of spinor algebras. Then this construction was extended to HS algebras in AdS_3 [47,48,49] and to $d = 4$ conformal HS algebras [50, 51] equivalent to the AdS_5 algebras of [7]. The $d = 7$ HS algebras [52] were also built in spinorial terms. Conformal HS conserved currents in any dimension, generating HS symmetries with the parameters carrying representations of the conformal algebra $o(d, 2)$ described by various rectangular two-row Young tableaux, were found in [53]. The realization of the conformal HS algebra h in any dimension in terms of a quotient of the universal enveloping algebra[10] $\mathcal{U}(o(d, 2))$ was given by Eastwood in [54]. Here we use the construction of the same algebra as given in [16], which is based on vector oscillator algebra (*i.e.* Weyl algebra).

5.1 Weyl algebras

The Weyl algebra A_{d+1} is the associative algebra generated by the oscillators $\hat{Y}_i^A$, where $i = 1, 2$ and $A = 0, 1, \ldots d$, satisfying the commutation relations

$$[\hat{Y}_i^A, \hat{Y}_j^B] = \epsilon_{ij}\, \eta^{AB}\,, \tag{5.2}$$

where $\epsilon_{ij} = -\epsilon_{j\,i}$ and $\epsilon_{12} = \epsilon^{12} = 1$. The invariant metrics $\eta_{AB} = \eta_{BA}$ and symplectic form ϵ^{ij} of $o(d-1,2)$ and $sp(2)$, respectively, are used to raise and lower indices in the usual manner $A^A = \eta^{AB} A_B$, $a^i = \epsilon^{ij} a_j$, $a_i = a^j \epsilon_{j\,i}$. The Weyl algebra A_{d+1} can be realized by taking as generators

$$\hat{Y}_1^A = \eta^{AB} \frac{\partial}{\partial X^B}\,, \quad \hat{Y}_2^A = X^A\,,$$

i.e. the Weyl algebra is realized as the algebra of differential operators acting on formal power series $\Phi(X)$ in the variable X^A. One can consider both complex ($A_{d+1}(\mathbb{C})$) and real Weyl ($A_{d+1}(\mathbb{R})$) algebras. One can also construct the (say, real) Weyl algebra $A_{d+1}(\mathbb{R})$ starting from the associative algebra $\mathbb{R} < \hat{Y}_i^A >$ freely generated by the variables $\hat{Y}_i^A$, *i.e.* spanned by all (real) linear combinations of all possible products of the variables $\hat{Y}_i^A$. The real Weyl algebra A_{d+1} is realized as the quotient of $\mathbb{R} < \hat{Y}_i^A >$ by the ideal made of all elements proportional to

$$\hat{Y}_i^A \hat{Y}_j^B - \hat{Y}_j^B \hat{Y}_i^A - \epsilon_{ij} \eta^{AB}\,.$$

In order to pick one representative of each equivalence class, we work with Weyl ordered operators. These are the operators completely symmetric under the exchange of $\hat{Y}_i^A$'s. The generic element of

[10] A universal enveloping algebra is defined as follows. Let $\mathcal{S}$ be the associative algebra that is freely generated by the elements of a Lie algebra s. Let $\mathcal{I}$ be the ideal of $\mathcal{S}$ generated by elements of the form $xy - yx - [x, y]$ ($x, y \in s$). The quotient $\mathcal{U}(s)= \mathcal{S}/\mathcal{I}$ is called the universal enveloping algebra of s.

A_{d+1} is then of the form

$$f(\hat{Y}) = \sum_{p=0}^{\infty} \phi^{i_1 \dots i_p}_{A_1 \dots A_p} \hat{Y}^{A_1}_{i_1} \dots \hat{Y}^{A_p}_{i_p} \,, \tag{5.3}$$

where $\phi^{i_1 \dots i_p}_{A_1 \dots A_p}$ is symmetric under the exchange $(i_k, A_k) \leftrightarrow (i_l, A_l)$. Equivalently, one can define basis elements $S^{A_1 \dots A_m \,, B_1 \dots B_n}$ that are completely symmetrized products of m $\hat{Y}^A_1$'s and n $\hat{Y}^B_2$'s (*e.g.* $S^{A,B} = \{\hat{Y}^A_1 \,, \hat{Y}^B_2\}$), and write the generic element as

$$f(\hat{Y}) = \sum_{m,n} f_{A_1 \dots A_m \,, B_1 \dots B_n} S^{A_1 \dots A_m \,, B_1 \dots B_n} \,, \tag{5.4}$$

where the coefficients $f_{A_1 \dots A_m \,, B_1 \dots B_n}$ are symmetric in the indices A_i and B_j.

The elements

$$T^{AB} = -T^{BA} = \frac{1}{4} \{\hat{Y}^{Ai} \,, \hat{Y}^B_i\} \tag{5.5}$$

satisfy the $o(d-1,2)$ algebra

$$[T^{AB}, T^{CD}] = \frac{1}{2} \Big(\eta^{BC} T^{AD} - \eta^{AC} T^{BD} - \eta^{BD} T^{AC} + \eta^{AD} T^{BC} \Big)$$

because of (5.2). When the Weyl algebra is realized as the algebra of differential operators, then $T^{AB} = X^{[A} \partial^{B]}$ generates rotations of $\mathbb{R}^{d-1,2}$ acting on a scalar $\Phi(X^A)$.

The operators

$$t_{ij} = t_{ji} = \frac{1}{2} \{\hat{Y}^A_i \,, \hat{Y}^B_j\} \eta_{AB} \tag{5.6}$$

generate $sp(2)$. The various bilinears T^{AB} and t_{ij} commute

$$[T^{AB}, t_{ij}] = 0 \,, \tag{5.7}$$

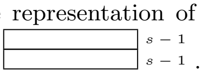
thus forming a Howe dual pair $o(d-1,2) \oplus sp(2)$.

5.2 Definition of the higher spin algebras

Let us consider the subalgebra $\mathcal{S}$ of elements $f(\hat{Y})$ of the complex Weyl algebra $A_{d+1}(\mathbb{C})$ that are invariant under $sp(2)$, *i.e.* $[f(\hat{Y}), t_{ij}] = 0$. Replacing $f(\hat{Y})$ by its Weyl symbol $f(Y)$, which is the ordinary function of commuting variables Y that has the same power series expansion as $f(\hat{Y})$ in the Weyl ordering, the $sp(2)$ invariance condition takes the form (a simple proof will be given in Section 11)

$$\Big(\epsilon_{k\,j} Y^A_i \frac{\partial}{\partial Y^A_k} + \epsilon_{k\,i} Y^A_j \frac{\partial}{\partial Y^A_k} \Big) f(Y^A_i) = 0 \,, \tag{5.8}$$

which is equivalent to (3.5) for $p = 2$. This condition implies that the coefficients $f_{A_1 \dots A_n \,, B_1 \dots B_m}$ vanish except when $n = m$, and the nonvanishing coefficients carry irreducible representations of $gl(d+1)$ corresponding to two-row rectangular Young tableaux. The $sp(2)$ invariance condition means in particular that (the symbol of) any element of $\mathcal{S}$ is an even function of Y^A_i. Let us note that the rôle of $sp(2)$ in our construction is reminiscent of that of $sp(2)$ in the conformal framework description of dynamical models (two-time physics) [55, 56].

However, the associative algebra $\mathcal{S}$ is not simple. It contains the ideal $\mathcal{I}$ spanned by the elements of $\mathcal{S}$ of the form $g = t_{ij}\, g^{ij} = g^{ij}\, t_{ij}$. Due to the definition of t_{ij} (5.6), all traces of two-row Young tableaux are contained in $\mathcal{I}$. As a result, the associative algebra $\mathcal{A} = \mathcal{S}/\mathcal{I}$ contains only all traceless two-row rectangular tableaux. Let us choose a basis $\{T_s\}$ of $\mathcal{A}$ where the elements T_s carry an irreducible representation of $o(d-1,2)$ characterized by a two-row Young tableau with $s-1$ columns: $T_s \sim$ 

.

Now consider the complex Lie algebra $h_{\mathbb{C}}$ obtained from the associative algebra $\mathcal{A}$ by taking the commutator as Lie bracket, the associativity property of $\mathcal{A}$ thereby translating into the Jacobi identity of $h_{\mathbb{C}}$. It admits several inequivalent real forms $h_{\mathbb{R}}$ such that $h_{\mathbb{C}} = h_{\mathbb{R}} \oplus i\, h_{\mathbb{R}}$. The particular real form that corresponds to a unitary HS theory is denoted by $hu(1|2{:}[d-1,2])$. This notation[11] refers to the Howe dual pair $sp(2) \oplus o(d-1,2)$ and to the fact that the related spin-1 Yang-Mills subalgebra is $u(1)$. The algebra $hu(1|2{:}[d-1,2])$ is spanned by the elements satisfying the following reality condition

$$(f(\hat{Y}))^\dagger = -f(\hat{Y}) \ , \tag{5.9}$$

where $\dagger$ is an involution[12] of the complex Weyl algebra defined by the relation

$$(\hat{Y}_i^A)^\dagger = i\hat{Y}_i^A \, . \tag{5.10}$$

Thanks to the use of the Weyl ordering prescription, reversing the order of the oscillators has no effect so that $(f(\hat{Y}))^\dagger = \bar{f}(i\hat{Y})$ where the bar means complex conjugation of the coefficients in the expansion (5.3)

$$\bar{f}(\hat{Y}) = \sum_{p=0}^{\infty} \bar{\phi}^{i_1\ldots i_p}_{A_1\ldots A_p}\, \hat{Y}^{A_1}_{i_1} \ldots \hat{Y}^{A_p}_{i_p} \ . \tag{5.11}$$

As a result, the reality condition (5.9) implies that the coefficients in front of the generators T_s (*i.e.* the basis elements $S^{A_1\ldots A_{s-1}\, ,B_1 \ldots B_{s-1}}$) with even and odd s are, respectively, real and pure imaginary. In particular, the spin-2 generator T^{AB} enters with a real coefficient.

What singles out $hu(1|2{:}[d-1,2])$ as the physically relevant real form is that it allows lowest weight unitary representations to be identified with the spaces of single particle states in the free HS theory [57]. For these unitary representations (5.9) becomes the antihermiticity property of the generators with $\dagger$ defined via a positive definite Hermitian form. As was argued in [44, 45, 46] for the similar problem in the case of $d = 4$ HS algebras, the real HS algebras that share this property are obtained by imposing the reality conditions based on an involution of the underlying complex associative algebra (*i.e.,* Weyl algebra).

Let us note that, as pointed out in [75] and will be demonstrated below in Section 10.1, the Lie algebra with the ideal $\mathcal{I}$ included, *i.e.,* resulting from the algebra $\mathcal{S}$ with the commutator as a Lie product and the reality condition (5.9), (5.10), underlies the off-mass-shell formulation of the HS gauge theory. We call this off-mass-shell HS algebra $hu_\infty(1|2{:}[d-1,2])$.[13]

5.3 Properties of the higher spin algebras

The real Lie algebra $hu(1|2{:}[d-1,2])$ is infinite-dimensional. It contains the space-time isometry algebra $o(d-1,2)$ as the subalgebra generated by T^{AB}. The basis elements T_s ($s \geqslant 1$) will be associated with a spin-s gauge field. In Section 14 we will show that $hu(1|2{:}[d-1,2])$ is indeed a global symmetry algebra of the AdS_d vacuum solution in the nonlinear HS gauge theory.

Taking two HS generators T_{s_1} and T_{s_2}, being homogeneous polynomials of degrees $2(s_1-1)$ and $2(s_2-1)$ in $\hat{Y}$, respectively, one obtains (modulo some coefficients)

$$[T_{s_1}\, , T_{s_2}] = \sum_{m=1}^{min(s_1,s_2)-1} T_{s_1+s_2-2m} = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2} \, . \tag{5.12}$$

[11] This notation was introduced in [57] instead of the more complicated one $hu(1/sp(2)[d-1,2])$ of [16].

[12] This means that $\dagger$ conjugates complex numbers, reverses the order of operators and squares to unity: $(\mu f)^\dagger = \bar{\mu} f^\dagger$, $(fg)^\dagger = g^\dagger f^\dagger$, $((f)^\dagger)^\dagger = f$. To be an involution of the Weyl algebra, $\dagger$ is required to leave invariant its defining relation (5.2). Any condition of the form (5.9) singles out a real linear space of elements that form a closed Lie algebra with respect to commutators.

[13] There exist also intermediate factor algebras $hu_P(1|2{:}[d-1,2])$ with the smaller ideals $\mathcal{I}_P$ factored out, where $\mathcal{I}_P$ is spanned by the elements of $\mathcal{S}$ of the form $t_{ij}P(c_2)g^{ij}$, where $c_2 = \frac{1}{2}t_{ij}t^{ij}$ is the quadratic Casimir operator of $sp(2)$ and $P(c_2)$ is some polynomial. The latter HS algebras are presumably of less importance because they should correspond to HS models with higher derivatives in the field equations even at the free field level.

Let us notice that the formula (5.12) is indeed consistent with the requirement (5.1). Furthermore, once a gauge field of spin $s>2$ appears, the HS symmetry algebra requires an infinite tower of HS gauge fields to be present, together with gravity. Indeed, the commutator $[T_s, T_s]$ of two spin-s generators gives rise to generators T_{2s-2}, corresponding to a gauge field of spin $s'=2s-2>s$, and also gives rise to generators T_2 of $o(d-1,2)$, corresponding to gravity fields. The spin-2 barrier separates theories with usual finite-dimensional lower-spin symmetries from those with infinite-dimensional HS symmetries. More precisely, the maximal finite-dimensional subalgebra of $hu(1|2{:}[d-1,2])$ is the direct sum: $u(1)\oplus o(d-1,2)$, where $u(1)$ is the center associated with the elements proportional to the unit. Another consequence of the commutation relations (5.12) is that even spin generators T_{2p} $(p\geqslant 1)$ span a proper subalgebra of the HS algebra, denoted as $ho(1|2{:}[d-1,2])$ in [57].

The general structure of the commutation relations (5.12) follows from the simple fact that the associative Weyl algebra possesses an antiautomorphism such that $\rho(f(\hat{Y}))=f(i\hat{Y})$ in the Weyl ordering. (The difference between ρ and $\dagger$ is that the former does not conjugate complex numbers.) It induces an automorphism τ of the Lie algebra $hu(1|2{:}[d-1,2])$ with

$$\tau(f(\hat{Y}))=-f(i\hat{Y})\,. \tag{5.13}$$

This automorphism is involutive in the HS algebra, *i.e.* $\tau^2=identity$, because $f(-\hat{Y})=f(\hat{Y})$. Therefore the algebra decomposes into subspaces of τ–odd and τ–even elements. Clearly, these are the subspaces of odd and even spins, respectively. This determines the general structure of the commutation relation (5.12), implying in particular that the even spin subspace forms the proper subalgebra $ho(1|2{:}[d-1,2])\subset hu(1|2{:}[d-1,2])$.

Alternatively, the commutation relation (5.12) can be obtained from the following reasoning. The oscillator commutation relation (5.2) contracts two $\hat{Y}$ variables and produces a tensor ϵ_{ij}. Thus, since the commutator of two polynomials is antisymmetric, only odd numbers of contractions can survive. A HS generator T_s is a polynomial of degree $2(s-1)$ in $\hat{Y}$ with the symmetries associated with the two-row Young tableau of length $s-1$. Computing the commutator $[T_{s_1},T_{s_2}]$, only odd numbers $2m-1$ of contractions survive $(m\geqslant 1)$ leading to polynomials of degree $2(s_1+s_2-2m-1)$ in $\hat{Y}$. They correspond to two-row rectangular Young tableaux[14] of length s_1+s_2-2m-1 that are associated to basis elements $T_{s_1+s_2-2m}$. The maximal number, say $2n-1$, of possible contractions is at most equal to the lowest polynomial degree in $\hat{Y}$ of the two generators. Actually, it must be one unit smaller since the numbers of surviving contractions are odd numbers while the polynomial degrees of the generators are even numbers. The lowest polynomial degree in Y of the two generators T_{s_1} and T_{s_2} is equal to $2\Big(min(s_1,s_2)-1\Big)$. Hence, $n=min(s_1,s_2)-1$. Consequently, the lowest possible polynomial degree of a basis element appearing on the right-hand-side of (5.12) is equal to $2(s_1+s_2-2n-1)=2(|s_1-s_2|+1)$. The corresponding generators are $T_{|s_1-s_2|+2}$.

The gauge fields of $hu(1|2{:}[d-1,2])$ are the components of the connection one-form

$$\omega(\hat{Y},x)=\sum_{s=1}^{\infty}dx^\mu\, i^{s-2}\omega_{\mu\ A_1\ldots A_{s-1}\,,B_1\ldots B_{s-1}}(x)\hat{Y}_1^{A_1}\ldots\hat{Y}_1^{A_{s-1}}\hat{Y}_2^{B_1}\ldots\hat{Y}_2^{B_{s-1}}\,. \tag{5.14}$$

They take values in the traceless two-row rectangular Young tableaux of $o(d-1,2)$. It is obvious from this formula why the basis elements T_s are associated to spin-s fields. The curvature and gauge transformations have the standard Yang-Mills form

$$R=d\omega+\omega^2\Big|_{\mathcal{I}\sim 0}\,,\qquad \delta\omega=D\epsilon\equiv d\epsilon+[\omega,\epsilon]\Big|_{\mathcal{I}\sim 0} \tag{5.15}$$

except that the product of two elements (5.14) with traceless coefficients is not necessarily traceless so that the ideal $\mathcal{I}$ has to be factored out in the end. More precisely, the products ω^2 and $[\omega,\epsilon]$ have to be represented as sums of elements of the algebra with traceless coefficients and others of the form $g_{ij}t^{ij}$

[14]Note that the formal tensor product of two two-row rectangular Young tableaux contains various Young tableaux having up to four rows. The property that only two-row Young tableaux appear in the commutator of HS generators is a consequence of the $sp(2)$ invariance condition.

(equivalently, taking into account the $sp(2)$ invariance condition, $t^{ij}\tilde{g}_{ij}$). The latter terms have then to be dropped out, and the resulting factorization is denoted by the symbol $\Big|_{\mathcal{I}\sim 0}$ (see also the discussion in Subsections 12.2 and 14.3).

The formalism here presented is equivalent [57] to the spinor formalism developed previously for lower dimensions, where the HS algebra was realized in terms of commuting spinor oscillators $\hat{y}^\alpha, \hat{\bar{y}}^{\dot{\alpha}}$ (see, for example, [3, 58] for reviews) as

$$[\hat{y}^\alpha, \hat{y}^\beta] = i\epsilon^{\alpha\beta}\ , \quad [\hat{\bar{y}}^{\dot{\alpha}}, \hat{\bar{y}}^{\dot{\beta}}] = i\epsilon^{\dot{\alpha}\dot{\beta}}\ , \quad [\hat{y}^\alpha, \hat{\bar{y}}^{\dot{\beta}}] = 0\ .$$

Though limited to $d = 3, 4$, the definition of the HS algebra with spinorial oscillators is simpler than that with vectorial oscillators Y_i^A, since the generators are automatically traceless (because $\hat{y}^\alpha\hat{y}_\alpha = \hat{\bar{y}}^{\dot{\alpha}}\hat{\bar{y}}_{\dot{\alpha}} = const$), and there is no ideal to be factored out. However, spinorial realizations of $d = 4$ conformal HS algebras [50, 51] (equivalent to the AdS_5 algebras [7]) and AdS_7 HS algebras [52] require the factorization of an ideal.

6 Free differential algebras and unfolded dynamics

Subsection 6.1 reviews some general definitions of the unfolded formulation of dynamical systems, a particular case of which are the HS field equations [59, 60]. The strategy of the unfolded formalism is presented in Subsection 6.2. It makes use of free differential algebras [61] in order to write consistent nonlinear dynamics. In more modern terms the fundamental underlying concept is L_∞ algebra [62].

6.1 Definition and examples of free differential algebras

Let us consider an arbitrary set of differential forms $W^\alpha \in \Omega^{p_\alpha}(\mathcal{M}^d)$ with degree $p_\alpha \geqslant 0$ (zero-forms are included), where α is an index enumerating various forms, which, generically, may range in the infinite set $1 \leqslant \alpha < \infty$.

Let $R^\alpha \in \Omega^{p_\alpha+1}(\mathcal{M}^d)$ be the generalized curvatures defined by the relations

$$R^\alpha = dW^\alpha + G^\alpha(W)\,, \tag{6.1}$$

where

$$G^\alpha(W^\beta) = \sum_{n=1}^{\infty} f^\alpha_{\beta_1\ldots\beta_n} W^{\beta_1}\ldots W^{\beta_n} \tag{6.2}$$

are some power series in W^β built with the aid of the exterior product of differential forms. The (anti)symmetry properties of the structure constants $f^\alpha_{\beta_1\ldots\beta_n}$ are such that $f^\alpha_{\beta_1\ldots\beta_n} \neq 0$ for $p_\alpha + 1 = \sum_{i=1}^n p_{\beta_i}$ and the permutation of any two indices β_i and β_j brings a factor of $(-1)^{p_{\beta_i}p_{\beta_j}}$ (in the case of bosonic fields, *i.e.* with no extra Grassmann grading in addition to that of the exterior algebra).

The choice of a function $G^\alpha(W^\beta)$ satisfying the generalized Jacobi identity

$$G^\beta \frac{\delta^L G^\alpha}{\delta W^\beta} \equiv 0 \tag{6.3}$$

(the derivative with respect to W^β is left) defines a free differential algebra[15] [61] introduced originally in the field-theoretical context in [64]. We emphasize that the property (6.3) is a condition on the function

[15] We remind the reader that a differential d is a Grassmann odd nilpotent derivation of degree one, *i.e.* it satisfies the (graded) Leibnitz rule and $d^2 = 0$. A differential algebra is a graded algebra endowed with a differential d. Actually, the "free differential algebras" (in physicist terminology) are more precisely christened "graded commutative free differential algebra" by mathematicians (this means that the algebra does not obey algebraic relations apart from graded commutativity). In the absence of zero-forms (which however play a key role in the unfolded dynamics construction) the structure of these algebras is classified by Sullivan [63].

$G^\alpha(W)$ to be satisfied identically for all W^β. It is equivalent to the following generalized Jacobi identity on the structure coefficients

$$\sum_{n=0}^{m}(n+1)f^{\gamma}_{[\beta_1\dots\beta_{m-n}}f^{\alpha}_{\gamma\beta_{m-n+1}\dots\beta_m\}}=0\,,\tag{6.4}$$

where the brackets $[\,\}$ denote an appropriate (anti)symmetrization of all indices β_i. Strictly speaking, the generalized Jacobi identities (6.4) have to be satisfied only at $p_\alpha < d$ for the case of a d-dimensional manifold $\mathcal{M}^d$ where any $d+1$-form is zero. We will call a free differential algebra *universal* if the generalized Jacobi identity is true for all values of indices, *i.e.,* independently of a particular value of space-time dimension. The HS free differential algebras discussed in this paper belong to the universal class. Note that every universal free differential algebra defines some L_∞ algebra[16].

The property (6.3) guarantees the generalized Bianchi identity

$$dR^\alpha = R^\beta\frac{\delta^L G^\alpha}{\delta W^\beta}\,,$$

which tells us that the differential equations on W^β

$$R^\alpha(W)=0\tag{6.5}$$

are consistent with $d^2=0$ and supercommutativity. Conversely, the property (6.3) is necessary for the consistency of the equation (6.5).

For universal free differential algebras one defines the gauge transformations as

$$\delta W^\alpha = d\varepsilon^\alpha - \varepsilon^\beta\frac{\delta^L G^\alpha}{\delta W^\beta}\,,\tag{6.6}$$

where $\varepsilon^\alpha(x)$ has form degree equal to $p_\alpha-1$ (so that zero-forms W^α do not give rise to any gauge parameter). With respect to these gauge transformations the generalized curvatures transform as

$$\delta R^\alpha = -R^\gamma\frac{\delta^L}{\delta W^\gamma}\left(\varepsilon^\beta\frac{\delta^L G^\alpha}{\delta W^\beta}\right)$$

due to the property (6.3). This implies the gauge invariance of the equations (6.5). Also, since the equations (6.5) are formulated entirely in terms of differential forms, they are explicitly general coordinate invariant.

Unfolding means reformulating the dynamics of a system into an equivalent system of the form (6.5), which, as is explained below, is always possible by virtue of introducing enough auxiliary fields. Note that, according to (6.1), in this approach the exterior differential of all fields is expressed in terms of the fields themselves. A nice property of the universal free differential algebras is that they allow an equivalent description of unfolded systems in larger (super)spaces simply by adding additional coordinates corresponding to a larger (super)space as was demonstrated for some particular examples in [51, 89, 17, 71].

Let h be a Lie (super)algebra, a basis of which is the set $\{T_\alpha\}$. Let $\omega=\omega^\alpha T_\alpha$ be a one-form taking values in h. If one chooses $G(\omega)=\omega^2\equiv\frac{1}{2}\omega^\alpha\omega^\beta[T_\alpha,T_\beta]$, then the equation (6.5) with $W=\omega$ is the zero-curvature equation $d\omega+\omega^2=0$. The relation (6.3) amounts to the usual Jacobi identity for the Lie (super)algebra h as is most obvious from (6.4) (or its super version). In the same way, (6.6) is the usual gauge transformation of the connection ω.

If the set W^α also contains some p-forms denoted by $\mathcal{C}^i$ (*e.g.* zero-forms) and if the functions G^i are linear in ω and $\mathcal{C}$,

$$G^i=\omega^\alpha(T_\alpha)^i{}_j\mathcal{C}^j\,,\tag{6.7}$$

[16] The minor difference is that a form degree p_α of W^α is fixed in a universal free differential algebra while W^α in L_∞ are treated as coordinates of a graded manifold. A universal free differential algebra can therefore be obtained from L_∞ algebra by an appropriate projection to specific form degrees.

then the relation (6.3) implies that the coefficients $(T_\alpha)^i{}_j$ define some matrices T_α forming a representation T of h, acting in a module V where the $\mathcal{C}^i$ take their values. The corresponding equation (6.5) is a covariant constancy condition $D_\omega \mathcal{C} = 0$, where $D_\omega \equiv d + \omega$ is the covariant derivative in the h-module V.

6.2 Unfolding strategy

From the previous considerations, one knows that the system of equations

$$d\omega_0 + \omega_0^2 = 0\,, \tag{6.8}$$

$$D_{\omega_0}\mathcal{C} = 0 \tag{6.9}$$

forms a free differential algebra. The first equation usually describes a background (for example Minkowski or AdS) along with some pure gauge modes. The connection one-form ω_0 takes value in some Lie algebra h. The second equation may describe nontrivial dynamics if $\mathcal{C}$ is a zero-form C that forms an infinite-dimensional h-module T appropriate to describe the space of all moduli of solutions (*i.e.*, the initial data). One can wonder how the set of equations (6.8) and

$$D_{\omega_0}C = 0 \tag{6.10}$$

could describe any dynamics, since it implies that (locally) the connection ω_0 is pure gauge and C is covariantly constant, so that

$$\begin{aligned} \omega_0(x) &= g^{-1}(x)\, dg(x)\,, && (6.11)\\ C(x) &= g^{-1}(x) \cdot \mathrm{C}\,, && (6.12) \end{aligned}$$

where $g(x)$ is some function of the position x taking values in the Lie group H associated with h (by exponentiation), C is a constant vector of the h-module T and the dot stands for the corresponding action of H on T. Since the gauge parameter $g(x)$ does not carry any physical degree of freedom, all physical information is contained in the value $C(x_0) = g^{-1}(x_0) \cdot \mathrm{C}$ of the zero-form $C(x)$ at a fixed point x_0 of space-time. But as one will see in Section 7, if the zero-form $C(x)$ somehow parametrizes all the derivatives of the original dynamical fields, then, supplemented with some algebraic constraints (that, in turn, single out an appropriate h-module), it can actually describe nontrivial dynamics. More precisely, the restrictions imposed on values of some zero-forms at a fixed point x_0 of space-time can lead to nontrivial dynamics if the set of zero-forms is rich enough to describe all space-time derivatives of the dynamical fields at a fixed point of space-time, provided that the constraints just single out those values of the derivatives that are compatible with the original dynamical equations. By knowing a solution (6.12) one knows all the derivatives of the dynamical fields compatible with the field equations and can therefore reconstruct these fields by analyticity in some neighborhood of x_0.

The p-forms with $p > 0$ contained in $\mathcal{C}$ (if any) are still pure gauge in these equations. As will be clear from the examples below, the meaning of the zero-forms C contained in $\mathcal{C}$ is that they describe all gauge invariant degrees of freedom (*e.g.* the spin-0 scalar field, the spin-1 Maxwell field strength, the spin-2 Weyl tensor, etc., and all their on-mass-shell nontrivial derivatives). When the gauge invariant zero-forms are identified with derivatives of the gauge fields which are $p > 0$ forms, this is expressed by a deformation of the equation (6.9)

$$D_{\omega_0}\mathcal{C} = P(\omega_0)\mathcal{C}\,, \tag{6.13}$$

where $P(\omega_0)$ is a linear operator (depending on ω_0 at least quadratically) acting on $\mathcal{C}$. The equations (6.8) and (6.13) are of course required to be consistent, *i.e.* to describe some free differential algebra, which is a deformation of (6.8) and (6.9). If the deformation is trivial, one can get rid of the terms on the right-hand-side of (6.13) by a field redefinition. The interesting case therefore is when the deformation is nontrivial. A useful criterium of whether the deformation (6.13) is trivial or not is given in terms of the σ_- cohomology in Section 9.

The next step is to interpret the equations (6.8) and (6.13) as resulting from the linearization of some nonlinear system of equations with

$$W = \omega_0 + \mathcal{C} \tag{6.14}$$

in which ω_0 is some fixed zero-order background field chosen to satisfy (6.8) while $\mathcal{C}$ describes first order fluctuations. Consistency of this identification however requires nonlinear corrections to the original linearized equations because the full covariant derivatives built of $W = \omega_0 + \mathcal{C}$ develop nonzero curvature due to the right hand side of (6.13). Finding these nonlinear corrections is equivalent to finding interactions.

This suggests the following strategy for the analysis of HS gauge theories:

1. One starts from a space-time with some symmetry algebra s (*e.g.* Poincaré or anti-de Sitter algebra) and a vacuum gravitational gauge field ω_0, which is a one-form taking values in s and satisfying the zero curvature equations (6.8).

2. One reformulates the field equations of a given free dynamical system in the "unfolded form" (6.13). This can always be done in principle (the general procedure is explained in Section 7). The only questions are: "how simple is the explicit formulation?" and "what are the modules T of s for which the unfolded equation (6.10) can be interpreted as a covariant constancy condition?".

3. One looks for a nonlinear free differential algebra such that (6.5) with (6.14) correctly reproduces the free field equations (6.8) and (6.13) at the linearized level. More precisely, one looks for some function $G(W)$ verifying (6.3) and the Taylor expansion of which around ω_0 is given by

$$G(W) = \omega_0^2 + \Big(\omega_0 - P(\omega_0)\Big)\mathcal{C} + O(\mathcal{C}^2)\,,$$

where $\omega_0\mathcal{C}$ denotes the action of ω_0 in the h-module $\mathcal{C}$ and the terms denoted by $O(\mathcal{C}^2)$ are at least quadratic in the fluctuation.

It is not *a priori* guaranteed that some nonlinear deformation exists at all. If not, this would mean that no consistent nonlinear equations exist. But if the deformation is found, then the problem is solved because the resulting equations are formally consistent, gauge invariant and generally coordinate invariant[17] as a consequence of the general properties of free differential algebras, and, by construction, they describe the correct dynamics at the free field level.

To find some nonlinear deformation, one has to address two related questions. The first one is "what is a relevant s-module T in which zero-forms that describe physical degrees of freedom in the model can take values?" and the second is "which infinite-dimensional (HS) extension h of s, in which one-form connections take their values, can act on the s-module T?". A natural candidate is a Lie algebra h constructed via commutators from the associative algebra $\mathcal{A}$

$$\mathcal{A} = \mathcal{U}(s)/Ann(T)\,,$$

where $\mathcal{U}(s)$ is the universal enveloping algebra of s while $Ann(T)$ is the annihilator, *i.e.* the ideal of $\mathcal{U}(s)$ spanned by the elements which trivialize on the module T. Of course, this strategy may be too naive in general because not all algebras can be symmetries of a consistent field-theoretical model and only some subalgebras of h resulting from this construction may allow a consistent nonlinear deformation. A useful criterium is the *admissibility condition* [66] which requires that there should be a unitary h-module which describes a list of quantum single-particle states corresponding to all HS gauge fields described in terms of the connections of h. If no such representation exists, there is no chance to find a nontrivial consistent (in particular, free of ghosts) theory that admits h as a symmetry of its most symmetric vacuum. In any case, $\mathcal{U}(s)$ is the reasonable starting point to look for a HS algebra[18]. It

[17] Note that any fixed choice of ω_0 breaks down the diffeomorphism invariance to a global symmetry of the vacuum solution. This is why the unfolding formulation works equally well both in the theories with fixed background field ω_0 and no manifest diffeomorphism invariance and those including gravity where ω_0 is a zero order part of the dynamical gravitational field.

[18] Based on somewhat different arguments, this idea was put forward by Fradkin and Linetsky in [65].

seems to be most appropriate, however, to search for conformal HS algebras. Indeed, the associative algebra $\mathcal{A}$ introduced in Section 5 is a quotient $\mathcal{A} = \mathcal{U}(s)/Ann(T)$, where T represents its conformal realization in $d-1$ dimensions [54, 57]. The related real Lie algebra h is $hu(1|2{:}[d-1,2])$. The space of single-particle quantum states of free massless HS fields of Section 8 provides a unitary module of $hu(1|2{:}[d-1,2])$ in which all massless completely symmetric representations of $o(d-1,2)$ appear just once [57].

7 Unfolding lower spins

The dynamics of any consistent system can in principle be rewritten in the unfolded form (6.5) by adding enough auxiliary variables [67]. This technique is explained in Subsection 7.1. Two particular examples of the general procedure are presented: the unfolding of the Klein-Gordon equation and the unfolding of gravity, in Subsections 7.2 and 7.3, respectively.

7.1 Unfolded dynamics

Let $\omega_0 = e_0^a\, P_a + \frac{1}{2}\omega_0^{ab} M_{ab}$ be a vacuum gravitational gauge field taking values in some space-time symmetry algebra s. Let $C^{(0)}(x)$ be a given space-time field satisfying some dynamical equations to be unfolded. Consider for simplicity the case where $C^{(0)}(x)$ is a zero-form. The general procedure of unfolding free field equations goes schematically as follows:

For a start, one writes the equation

$$D_0^L C^{(0)} = e_0^a \; C_a^{(1)} \,, \tag{7.1}$$

where D_0^L is the covariant Lorentz derivative and the field $C_a^{(1)}$ is auxiliary. Next, one checks whether the original field equations for $C^{(0)}$ impose any restrictions on the first derivatives of $C^{(0)}$. More precisely, some part of $D^L_{0\,,\,\mu} C^{(0)}$ might vanish on-mass-shell (*e.g.* for Dirac spinors). These restrictions in turn impose some restrictions on the auxiliary fields $C_a^{(1)}$. If these constraints are satisfied by $C_a^{(1)}$, then these fields parametrize all on-mass-shell nontrivial components of first derivatives.

Then, one writes for these first level auxiliary fields an equation similar to (7.1)

$$D_0^L C_a^{(1)} = e_0^b \; C_{a,b}^{(2)} \,, \tag{7.2}$$

where the new fields $C_{a,b}^{(2)}$ parametrize the second derivatives of $C^{(0)}$. Once again one checks (taking into account the Bianchi identities) which components of the second level fields $C_{a,b}^{(2)}$ are nonvanishing provided that the original equations of motion are satisfied.

This process continues indefinitely, leading to a chain of equations having the form of some covariant constancy condition for the chain of fields $C^{(m)}_{a_1,a_2,\dots,a_m}$ ($m \in \mathbb{N}$) parametrizing all on-mass-shell nontrivial derivatives of the original dynamical field. By construction, this leads to a particular unfolded equation (6.5) with G^i in (6.1) given by (6.7). As explained in Section 6.1, this means that the set of fields realizes some module T of the space-time symmetry algebra s. In other words, the fields $C^{(m)}_{a_1,a_2,\dots,a_m}$ are the components of a single field C living in the infinite-dimensional s–module T. Then the infinite chain of equations can be rewritten as a single covariant constancy condition $D_0 C = 0$, where D_0 is the s-covariant derivative in T.

7.2 The example of the scalar field

As a preliminary to the gravity example considered in the next subsection, the simplest field-theoretical case of unfolding is reviewed, *i.e.* the unfolding of a massless scalar field, which was first described in [67].

For simplicity, for the remaining of Section 7, we will consider the flat space-time background. The Minkowski solution can be written as

$$\omega_0 = dx^\mu \delta_\mu^a P_a \tag{7.3}$$

i.e. the flat frame is $(e_0)_\mu^a = \delta_\mu^a$ and the Lorentz connection vanishes. The equation (7.3) corresponds to the "pure gauge" solution (6.11) with

$$g(x) = \exp(x^\mu\, \delta_\mu^a\, P_a)\,, \tag{7.4}$$

where the space-time Lie algebra s is identified with the Poincaré algebra $iso(d-1,1)$. Though the vacuum ω_0 solution has a pure gauge form (7.4), this solution cannot be gauged away because of the constraint $\mathrm{rank}(e_0) = d$ (see Section 2.1).

The "unfolding" of the massless Klein-Gordon equation

$$\Box C(x) = 0 \tag{7.5}$$

is relatively easy to work out, so we give directly the final result and we comment about how it is obtained afterwards.

To describe the dynamics of the spin-0 massless field $C(x)$, let us introduce the infinite collection of zero-forms $C_{a_1\dots a_n}(x)$ $(n = 0, 1, 2, \dots)$ that are completely symmetric traceless tensors

$$C_{a_1\dots a_n} = C_{\{a_1\dots a_n\}}\,, \quad \eta^{bc} C_{bca_3\dots a_n} = 0\,. \tag{7.6}$$

The "unfolded" version of the Klein-Gordon equation (7.5) has the form of the following infinite chain of equations

$$dC_{a_1\dots a_n} = e_0^b C_{a_1\dots a_n b} \quad (n = 0, 1, \dots)\,, \tag{7.7}$$

where we have used the opportunity to replace the Lorentz covariant derivative D_0^L by the ordinary exterior derivative d. It is easy to see that this system is formally consistent because applying d on both sides of (7.7) does not lead to any new condition,

$$d^2 C_{a_1\dots a_n} \;=\; -\, e_0^b\, dC_{a_1\dots a_n b} \;=\; -\, e_0^b\, e_0^c\, C_{a_1\dots a_n bc} \;=\; 0 \quad (n = 0, 1, \dots)\,,$$

since $e_0^b e_0^c = -e_0^c e_0^b$ because e_0^b is a one-form. As we know from Section 6.1, this property implies that the space T of zero-forms $C_{a_1\dots a_n}(x)$ spans some representation of the Poincaré algebra $iso(d-1,1)$. In other words, T is an infinite-dimensional $iso(d-1,1)$-module[19].

To show that this system of equations is indeed equivalent to the free massless field equation (7.5), let us identify the scalar field $C(x)$ with the member of the family of zero-forms $C_{a_1\dots a_n}(x)$ at $n = 0$. Then the first two equations of the system (7.7) read

$$\partial_\nu C = C_\nu\,,$$

$$\partial_\nu C_\mu = C_{\mu\nu}\,,$$

where we have identified the world and tangent indices via $(e_0)_\mu^a = \delta_\mu^a$. The first of these equations just tells us that C_ν is the first derivative of C. The second one tells us that $C_{\nu\mu}$ is the second derivative of C. However, because of the tracelessness condition (7.6) it imposes the Klein-Gordon equation (7.5). It is easy to see that all other equations in (7.7) express highest tensors in terms of the higher-order derivatives

$$C_{\nu_1\dots\nu_n} = \partial_{\nu_1} \dots \partial_{\nu_n} C \tag{7.8}$$

and impose no new conditions on C. The tracelessness conditions (7.6) are all satisfied once the Klein-Gordon equation is true. From this formula it is clear that the meaning of the zero-forms $C_{\nu_1\dots\nu_n}$ is that they form a basis in the space of all on-mass-shell nontrivial derivatives of the dynamical field $C(x)$ (including the derivative of order zero which is the field $C(x)$ itself).

[19] Strictly speaking, to apply the general argument of Section 6.1 one has to check that the equation remains consistent for any flat connection in $iso(d-1,1)$. It is not hard to see that this is true indeed.

Let us note that the system (7.7) without the constraints (7.6), which was originally considered in [68], remains formally consistent but is dynamically empty just expressing all highest tensors in terms of derivatives of C according to (7.8). This simple example illustrates how algebraic constraints like tracelessness of a tensor can be equivalent to dynamical equations.

The above consideration can be simplified further by means of introducing the auxiliary coordinate u^a and the generating function

$$C(x,u) = \sum_{n=0}^{\infty} \frac{1}{n\,!} C_{a_1\ldots a_n}(x) u^{a_1} \ldots u^{a_n}$$

with the convention that

$$C(x,0) = C(x)\,.$$

This generating function accounts for all tensors $C_{a_1\ldots a_n}$ provided that the tracelessness condition is imposed, which in these terms implies that

$$\Box_u C(x,u) \equiv \frac{\partial}{\partial u^a}\frac{\partial}{\partial u_a} C = 0\,. \tag{7.9}$$

In other words, the $iso(d-1,1)$-module T is realized as the space of formal harmonic power series in u^a. The equations (7.7) then acquire the simple form

$$\frac{\partial}{\partial x^\mu} C(x,u) = \delta_\mu^a \frac{\partial}{\partial u^a} C(x,u)\,. \tag{7.10}$$

From this realization one concludes that the translation generators in the infinite-dimensional module T of the Poincaré algebra are realized as translations in the u–space, *i.e.*

$$P_a = -\frac{\partial}{\partial u^a}\,,$$

for which the equation (7.10) reads as a covariant constancy condition (6.9)

$$dC(x,u) + e_0^a P_a C(x,u) = 0\,. \tag{7.11}$$

One can find a general solution of the equation (7.11) in the form

$$C(x,u) = C(x+u,0) = C(0,x+u) \tag{7.12}$$

from which it follows in particular that

$$C(x) \equiv C(x,0) = C(0,x) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{\nu_1\ldots\nu_n}(0) x^{\nu_1} \ldots x^{\nu_n}\,. \tag{7.13}$$

From (7.6) and (7.8) one can see that this is indeed the Taylor expansion for any solution of the Klein-Gordon equation which is analytic at $x_0 = 0$. Moreover the general solution (7.12) expresses the covariant constancy of the vector $C(x,u)$ of the module T,

$$C(x,u) = C(0,x+u) = \exp(-x^\mu\, \delta_\mu^a\, P_a) C(0,u)\,.$$

This is a particular realization of the pure gauge solution (6.12) with the gauge function $g(x)$ of the form (7.4) and $\mathrm{C} = C(0,u)$.

The example of a free scalar field is so simple that one might think that the unfolding procedure is always like a trivial mapping of the original equation (7.5) to the equivalent one (7.9) in terms of additional variables. This is not true, however, for the less trivial cases of dynamical systems in nontrivial backgrounds and, especially, nonlinear systems. The situation here is analogous to that in the Fedosov quantization prescription [69] which reduces the nontrivial problem of quantization in a curved

background to the standard problem of quantization of the flat phase space, that, of course, becomes an identity when the ambient space is flat itself. It is worth to mention that this parallelism is not accidental because, as one can easily see, the Fedosov quantization prescription provides a particular case of the general unfolding approach [59] in the dynamically empty situation (*i.e.*, with no dynamical equations imposed).

7.3 The example of gravity

The set of fields in Einstein-Cartan's formulation of gravity is composed of the frame field e^a_μ and the Lorentz connection ω^{ab}_μ. One supposes that the torsion constraint $T_a = 0$ is satisfied, in order to express the Lorentz connection in terms of the frame field. The Lorentz curvature can be expressed as $R^{ab} = e_c e_d\, R^{[cd]\,;\,[ab]}$, where $R^{ab\,;\,cd}$ is a rank four tensor with indices in the tangent space and which is antisymmetric both in ab and in cd, having the symmetries of the tensor product $\young(a,b) \otimes \young(c,d)$. The algebraic Bianchi identity $e_b R^{ab} = 0$, which follows from the zero torsion constraint, imposes that the tensor $R^{ab\,;\,cd}$ possesses the symmetries of the Riemann tensor, *i.e.* $R^{[ab\,;\,c]d} = 0$. More precisely, it carries an irreducible representation of $GL(d)$ characterized by the Young tableau $\young(ac,bd)$ in the antisymmetric basis. The vacuum Einstein equations state that this tensor is traceless, so that it is actually irreducible under the pseudo-orthogonal group $O(d-1,1)$ on-mass-shell. In other words, the Riemann tensor is equal on-mass-shell to the Weyl tensor.

For HS generalization, it is more convenient to use the symmetric basis. In this convention, the Einstein equations can be written as

$$T^a = 0\,, \qquad R^{ab} \,=\, e_c\, e_d\, C^{ac,\,bd}\,, \tag{7.14}$$

where the zero-form $C^{ac,\,bd}$ is the Weyl tensor in the symmetric basis. More precisely, the tensor $C^{ac,bd}$ is symmetric in the pairs ac and bd and it satisfies the algebraic identities

$$C^{\{ac,\,b\}d} = 0\,, \qquad \eta_{ac} C^{ac,\,bd} = 0\,.$$

Let us now start the unfolding of linearized gravity around the Minkowski background described by a frame one-form e^a_0 and Lorentz covariant derivative D^L_0. The linearization of the second equation of (7.14) is

$$R^{ab}_1 \,=\, e_{0\ c}\, e_{0\ d}\, C^{ac,\,bd}\,, \tag{7.15}$$

where R^{ab}_1 is the linearized Riemann tensor. This equation is a particular case of the equation (6.13). What is lacking at this stage is the equations containing the differential of the Weyl zero-form $C^{ac,\,bd}$. Since we do not want to impose any additional dynamical restrictions on the system, the only restrictions on the derivatives of the Weyl zero-form $C^{ac,\,bd}$ may result from the Bianchi identities applied to (7.15).

A priori, the first Lorentz covariant derivative of the Weyl tensor is a rank five tensor in the following representation

$$\yng(1) \otimes \yng(2,2) = \yng(2,2,1) \oplus \yng(3,2) \tag{7.16}$$

decomposed according to irreducible representations of $gl(d)$. Since the Weyl tensor is traceless, the right hand side of (7.16) contains only one nontrivial trace, that is for traceless tensors we have the $o(d-1,1)$ Young decomposition by adding a three cell hook tableau, *i.e.*

$$\yng(1) \otimes \yng(2,2) = \yng(2,2,1) \oplus \yng(3,2) \oplus \yng(2,1) \tag{7.17}$$

The linearized Bianchi identity $D_0^L R_1^{ab} = 0$ leads to

$$e_{0\,c} e_{0\,d}\, D_0^L C^{ac,\,bd} = 0\,. \tag{7.18}$$

The components of the left-hand-side written in the basis $dx^\mu dx^\nu dx^\rho$ have the symmetry property corresponding to the tableau

$$\begin{array}{|c|c|}\hline \mu & a \\ \hline \nu & b \\ \hline \rho & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \sim D^L_{0\,[\rho} C^{a}{}_{\mu}{}^{,b}{}_{\nu]}\,,$$

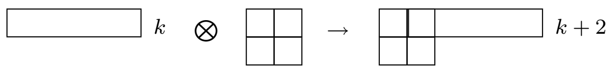
which also contains the single trace part with the symmetry properties of the three-cell hook tableau.

Therefore the consistency condition (7.18) says that in the decomposition (7.17) of the Lorentz covariant derivative of the Weyl tensor, the first and third terms vanish and the second term is traceless and otherwise arbitrary. Let $C^{abf,\,cd}$ be the traceless tensor corresponding to the second term in the decomposition (7.16) of the Lorentz covariant derivative of the Weyl tensor. This is equivalent to say that

$$D_0^L C^{ac,\,bd} \;=\; e_{0\,f}\,(2C^{acf,\,bd} + C^{acb,\,df} + C^{acd,\,bf})\,, \tag{7.19}$$

where the right hand side is fixed by the Young symmetry properties of the left hand side modulo an overall normalization coefficient. This equation looks like the first step (7.1) of the unfolding procedure. $C^{acf,\,bd}$ is irreducible under $o(d-1,1)$.

One should now perform the second step of the general unfolding scheme and write the analogue of (7.2). This process goes on indefinitely. To summarize the procedure, one can analyze the decomposition of the k-th Lorentz covariant derivatives (with respect to the Minkowski vacuum background, so they commute) of the Weyl tensor $C^{ac,\,bd}$. Taking into account the Bianchi identity, the decomposition goes as follows

 (7.20)

As a result, one obtains

$$D_0^L C^{a_1\dots a_{k+2},\,b_1b_2} \;=\; e_{0\,c}\Big(\,(k+2)\,C^{a_1\dots a_{k+2}c,\,b_1b_2} + C^{a_1\dots a_{k+2}b_1,\,b_2c} + C^{a_1\dots a_{k+2}b_2,\,b_1c}\Big)\,,$$
$$(0 \leqslant k \leqslant \infty)\,, \tag{7.21}$$

where the fields $C^{a_1\dots a_{k+2},\,b_1b_2}$ are in the irreducible representation of $o(d-1,1)$ characterized by the traceless two-row Young tableau on the right hand side of (7.20), *i.e.*

$$C^{\{a_1\dots a_{k+2},\,b_1\}b_2} = 0\,, \quad \eta_{a_1a_2} C^{a_1a_2\dots a_{k+2},\,b_1b_2} = 0\,.$$

Note that, as expected, the system (7.21) is consistent with $(D_0^L)^2 C^{a_1\dots a_{k+2},\,b_1b_2} = 0$.

Analogously to the spin-0 case, the meaning of the zero-forms $C^{a_1\dots a_{k+2},\,b_1b_2}$ is that they form a basis in the space of all on-mass-shell nontrivial gauge invariant combinations of the derivatives of the spin-2 gauge field.

In order to extend this analysis to nonlinear gravity, one replaces the background derivative D_0^L and frame field e_0^a by the full Lorentz covariant derivative D^L and dynamical frame e^a satisfying the zero torsion condition $D^L e^a = 0$ and

$$D^L D^L = R\,, \tag{7.22}$$

where R is the Riemann tensor taking values in the adjoint representation of the Lorentz algebra. The unfolding procedure goes the same way up to the equation (7.19) but needs nonlinear corrections starting from the next step. The reason is that Bianchi identities for (7.19) and analogous higher equations give rise to terms nonlinear in C via (7.22) and (7.14). All terms of second order in C in the nonlinear deformation of (7.21) were obtained in [60] for the case of four dimensions. The problem of unfolding nonlinear gravity in all orders remains unsolved.

8 Free massless equations for any spin

In order to follow the strategy exposed in Subsection 6.2 and generalize the example of gravity treated along these lines in Subsection 7.3, we shall start by writing unfolded HS field equations in terms of the linearized HS curvatures (4.4). This result is christened the "central on-mass-shell theorem". It was originally obtained in [40, 59] for the case of $d = 4$ and then extended to any d in [41, 30]. That these HS equations of motion indeed reproduce the correct physical degrees of freedom will be shown later in Section 10, via a cohomological approach explained in Section 9.

8.1 Connection one-form sector

The linearized curvatures $R_1^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ were defined by (4.4). They decompose into the linearized curvatures with Lorentz (*i.e.* V^A transverse) fibre indices which have the symmetry properties associated with the two-row traceless Young tableau $\begin{array}{l}\boxed{}^{\,s-1}\\ \boxed{}_{\,t}\end{array}$. It is convenient to use the standard gauge $V^A = \delta^A_{\hat d}$ (from now on we normalize V to unity). In the Lorentz basis, the linearized HS curvatures have the form

$$R_1^{a_1\dots a_{s-1},\,b_1\dots b_t} \;=\; D_0^L\,\omega^{a_1\dots a_{s-1},\,b_1\dots b_t} \;+\; e_{0\,c}\,\omega^{a_1\dots a_{s-1},\,b_1\dots b_t c} \;+\; O(\Lambda)\,. \tag{8.1}$$

For simplicity, in this section we discard the complicated Λ–dependent terms which do not affect the general analysis, *i.e.* we present explicitly the flat-space-time part of the linearized HS curvatures. It is important to note however that the Λ–dependent terms in (8.1) contain only the field $\omega^{a_1\dots a_{s-1},\,b_1\dots b_{t-1}}$ which carries one index less than the linearized HS curvatures. The explicit form of the Λ–dependent terms is given in [41].

For $t = 0$, these curvatures generalize the torsion of gravity, while for $t > 0$ the curvature corresponds to the Riemann tensor. In particular, as we will demonstrate in Section 10, the analogues of the Ricci tensor and scalar curvature are contained in the curvatures with $t = 1$ while the HS analog of the Weyl tensor is contained in the curvatures with $t = s-1$. (For the case of $s = 2$ they combine into the level $t = 1$ traceful Riemann tensor.)

The first on-mass-shell theorem states that the following free field equations in Minkowski or $(A)dS$ space-time

$$R_1^{a_1\dots a_{s-1},\,b_1\dots b_t} \;=\; \delta_{t,\,s-1}\;\; e_{0\,c}\,e_{0\,d}\;C^{a_1\dots a_{s-1}c,\,b_1\dots b_{s-1}d}\,, \qquad (0 \leqslant t \leqslant s-1) \tag{8.2}$$

properly describe completely symmetric gauge fields of generic spin $s \geqslant 2$. This means that they are equivalent to the proper generalization of the $d = 4$ Fronsdal equations of motion to any dimension, supplemented with certain algebraic constraints on the auxiliary HS connections which express the latter via derivatives of the dynamical HS fields. The zero-form $C^{a_1\dots a_s,\,b_1\dots b_s}$ is the spin-s Weyl-like tensor. It is irreducible under $o(d-1,1)$ and is characterized by a rectangular two-row Young tableau $\begin{array}{l}\boxed{}^{\,s}\\ \boxed{}_{\,s}\end{array}$. The field equations generalize (7.15) of linearized gravity. The equations of motion put to zero all curvatures with $t \neq s-1$ and require $C^{a_1\dots a_s,\,b_1\dots b_s}$ to be traceless.

8.2 Weyl zero-form sector

Note that the equations (8.2) result from the first step in the unfolding of the Fronsdal equations.[20] The analysis of the Bianchi identities of (8.2) works for any spin $s \geqslant 2$ in a way analogous to gravity. The

[20] Actually, the action and equations of motion for totally symmetric massless HS fields in AdS_d with $d > 4$ were originally obtained in [41] in the frame-like formalism.

final result is the following equation [30], which presents itself like a covariant constancy condition

$$
\begin{aligned}
0=\widetilde{D}_0 C^{a_1\dots a_{s+k},\,b_1\dots b_s} \quad\equiv\quad & D_0^L C^{a_1\dots a_{s+k},\,b_1\dots b_s} \\
& - e_{0\,c}\Big((k+2)C^{a_1\dots a_{s+k}c,\,b_1\dots b_s} \;+\; s\,C^{a_1\dots a_{s+k}\{b_1,\,b_2\dots b_s\}c}\Big)+O(\Lambda)\,,\\
& (0\leqslant k\leqslant\infty)\,,
\end{aligned}
\tag{8.3}
$$

where $C^{a_1\dots a_{s+k},\,b_1\dots b_s}$ are $o(d-1,1)$ irreducible (*i.e.*, traceless) tensors characterized by the Young tableaux $s+k$ s They describe on-mass-shell nontrivial k-th derivatives of the spin-s Weyl-like tensor, thus forming a basis in the space of gauge invariant combinations of $(s+k)$-th derivatives of a spin-s HS gauge field. The system (8.3) is the generalization of the spin-0 system (7.7) and the spin-2 system (7.21) to arbitrary spin and to AdS background (the explicit form of the Λ–dependent terms is given in [30]). Let us stress that for $s\geqslant 2$ the infinite system of equations (8.3) is a consequence of (8.2) by the Bianchi identity. For $s=0$ and $s=1$, the system (8.3) contains the dynamical Klein-Gordon and Maxwell equations, respectively. Note that (8.2) makes no sense for $s=0$ because there is no spin-0 gauge potential while (8.3) with $s=0$ reproduces the unfolded spin-0 equation (7.7) and its AdS generalization. For the spin-1 case, (8.2) only gives a definition of the spin-1 Maxwell field strength $C^{a,b}=-C^{b,a}$ in terms of the potential ω_μ. The dynamical equations for spin-1, *i.e.* Maxwell equations, are contained in (8.3). The fields $C^{a_1\dots a_{k+1},\,b}$, characterized by the Lorentz irreducible (*i.e.* traceless) two-row Young tableaux with one cell in the second row, form a basis in the space of on-mass-shell nontrivial derivatives of the Maxwell tensor $C^{a,\,b}$.

It is clear that the complete set of zero-forms $C^{a_1\dots a_{s+k},\,b_1\dots b_s}\sim$ $s+k$ s covers the set of all two-row Young tableaux. This suggests that the Weyl-like zero-forms take values in the linear space of $hu(1|2{:}[d-1,2])$, which obviously forms an $o(d-1,1)$-(*i.e.* Lorentz) module. Following Sections 6.1 and 6.2, one expects that the zero-forms belong to an $o(d-1,2)$-module T. But the idea to use the adjoint representation of $hu(1|2{:}[d-1,2])$ does not work because, according to the commutation relation (5.1), the commutator of the background gravity connection $\omega_0=\omega_0^{AB}T_{AB}$ with a generator of $hu(1|2{:}[d-1,2])$ preserves the rank of the generator, while the covariant derivative $\widetilde{D}_0$ in (8.3) acts on the infinite set of Lorentz tensors of infinitely increasing ranks. Fortunately, the appropriate representation only requires a slight modification compared to the adjoint representation. As will be explained in Section 12, the zero-forms C belong to the so-called "twisted adjoint representation".

Since k goes from zero to infinity for any fixed s in (8.3), in agreement with the general arguments of Section 6, each irreducible spin-s submodule of the twisted adjoint representation is infinite-dimensional. This means that, in the unfolded formulation, the dynamics of any fixed spin-s field is described in terms of an infinite set of fields related by the first-order unfolded equations. Of course, to make it possible to describe a field-theoretical dynamical system with an infinite number of degrees of freedom, the set of auxiliary zero-forms associated with all gauge invariant combinations of derivatives of dynamical fields should be infinite. Let us note that the right-hand-side of the equation (8.2) is a particular realization of the deformation terms (6.13) in free differential algebras.

The system of equations (8.2)-(8.3) provides the unfolded form of the free equations of motion for completely symmetric massless fields of all spins in any dimension. This fact is referred to as "central on-mass-shell theorem" because it plays a distinguished role in various respects. The idea of the proof will be explained in Section 10. The proof is based on a very general cohomological reformulation of the problem, which is reviewed in Section 9.

9 Dynamical content via σ_- cohomology

In this section, we perform a very general analysis of equations of motion of the form

$$\widehat{D}_0\mathcal{C}=0\ , \tag{9.1}$$

via a cohomological reformulation [68, 51, 17, 71] of the problem[21]. It will be applied to the HS context in the next section.

9.1 General properties of σ_-

Let us introduce the number of Lorentz indices as a grading G of the space of tensors with fiber (tangent) Lorentz indices. In the unfolded HS equations the background covariant derivative $\widehat{D}_0$ decomposes as the sum

$$\widehat{D}_0 = \sigma_- + D_0^L + \sigma_+ \,, \tag{9.2}$$

where the operator $\sigma_\pm$ modifies the rank of a Lorentz tensor by ± 1 and the background Lorentz covariant derivative D_0^L does not change it.

In the present consideration we do not fix the module V on which $\widehat{D}$ acts and, in (9.1), $\mathcal{C}$ denotes a set of differential forms taking values in V. In the context of the HS theory, the interesting cases are when $\widehat{D}$ acts either on the adjoint representation or on the twisted adjoint representation of the HS algebra. For example, the covariant constancy condition (8.3) takes the form

$$\widetilde{D}_0 C = (D_0^L + \sigma_- + \sigma_+)\, C = 0 \,,$$

where σ_+ denotes the Λ–dependent terms. The cohomological classification exposed here only assumes the following abstract properties:

(i) The grading operator G is diagonalizable in the vector space V and it possesses a spectrum bounded from below.

(ii) The grading properties of the Grassmann odd operators D_0^L and σ_- are summarized in the commutation relations

$$[G, D_0^L] = 0 \,, \qquad [G, \sigma_-] = -\sigma_- \,.$$

The operator σ_+ is a sum of operators of strictly positive grade. (In HS applications σ_+ has grade one, *i.e.* $[G, \sigma_+] = \sigma_+$, but this is not essential for the general analysis.)

(iii) The operator σ_- acts vertically in the fibre V, *i.e.* it does not act on space-time coordinates. (In HS models, only the operator D_0^L acts nontrivially on the space-time coordinates (differentiates).)

(iv) The background covariant derivative $\widehat{D}_0$ defined by (9.2) is nilpotent. The graded decomposition of the nilpotency equation $(\widehat{D}_0)^2 = 0$ gives the following identities

$$(\sigma_-)^2 = 0 \,, \quad D_0^L\sigma_- + \sigma_- D_0^L = 0 \,, \quad (D_0^L)^2 + \sigma_+\sigma_- + \sigma_-\sigma_+ + D_0^L\sigma_+ + \sigma_+ D_0^L + (\sigma_+)^2 = 0 \,. \tag{9.3}$$

If σ_+ has definite grade $+1$ (as is the case in the HS theories under consideration) the last relation is equivalent to the three conditions $(\sigma_+)^2 = 0 \,, \quad D_0^L\sigma_+ + \sigma_+ D_0^L = 0 \,, \quad (D_0^L)^2 + \sigma_+\sigma_- + \sigma_-\sigma_+ = 0$.

An important property is the nilpotency of σ_-. The point is that the analysis of Bianchi identities (as was done in details in Section 7.3 for gravity) is, in fact, equivalent to the analysis of the cohomology of σ_-, that is

$$H(\sigma_-) \equiv \frac{Ker(\sigma_-)}{Im(\sigma_-)} \,.$$

9.2 Cohomological classification of the dynamical content

Let $\mathcal{C}$ denote a differential form of degree p taking values in V, that is an element of the complex $V \otimes \Omega^p(\mathcal{M}^d)$. The field equation (9.1) is invariant under the gauge transformations

$$\delta\mathcal{C} = \widehat{D}_0 \varepsilon \,, \tag{9.4}$$

[21] Note that a similar approach was applied in a recent paper [72] in the context of $BRST$ formalism.

since $\widehat{D}_0$ is nilpotent by the hypothesis (iv). The gauge parameter ε is a $(p-1)$-form. These gauge transformations contain both differential gauge transformations (like linearized diffeomorphisms) and Stueckelberg gauge symmetries (like linearized local Lorentz transformations[22]).

The following terminology will be used. By *dynamical field*, we mean a field that is not expressed as derivatives of something else by field equations (*e.g.* the frame field in gravity or a frame-like HS one-form field $\omega_\mu{}^{a_1\ldots a_{s-1}}$). The fields that are expressed by virtue of the field equations as derivatives of the dynamical fields modulo Stueckelberg gauge symmeries are referred to as *auxiliary fields* (*e.g.* the Lorentz connection in gravity or its HS analogues $\omega_\mu{}^{a_1\ldots a_{s-1},\, b_1\ldots b_t}$ with $t>0$). A field that is neither auxiliary nor pure gauge by Stueckelberg gauge symmeries is said to be a *nontrivial dynamical field* (*e.g.* the metric tensor or the metric-like gauge fields of Fronsdal's approach).

Let $\mathcal{C}(x)$ be an element of the complex $V\otimes\Omega^p(\mathcal{M}^d)$ that satisfies the dynamical equation (9.1). Under the hypotheses (i)-(iv) one can prove the following propositions [68,51] (see also [17,71]):

A. Nontrivial dynamical fields $\mathcal{C}$ are nonvanishing elements of $H^p(\sigma_-)$.

B. Differential gauge symmetry parameters ε are classified by $H^{p-1}(\sigma_-)$.

C. Inequivalent differential field equations on the nontrivial dynamical fields contained in $\widehat{D}_0\mathcal{C}=0$ are in one-to-one correspondence with representatives of $H^{p+1}(\sigma_-)$.

Proof of A: The first claim is almost obvious. Indeed, let us decompose the field $\mathcal{C}$ according to the grade G:

$$\mathcal{C}=\sum_{n=0}\mathcal{C}_n\,,\qquad G\mathcal{C}_n=n\,\mathcal{C}_n\,,\qquad (n=0,1,2,\ldots)\,.$$

The field equation (9.1) thus decomposes as

$$\widehat{D}_0\mathcal{C}|_{\,n-1}=\sigma_-\mathcal{C}_n+D_0^L\mathcal{C}_{n-1}+\Big(\sigma_+\sum_{m\leqslant n-2}\mathcal{C}_m\Big)\Big|_{n-1}=0\,. \tag{9.5}$$

By a straightforward induction on $n=1,2,\ldots$, one can convince oneself that all fields $\mathcal{C}_n$ that contribute to the first term of the right hand side of the equation (9.5) are thereby expressed in terms of derivatives of lower grade (*i.e.* $<n$) fields, hence they are auxiliary[23]. As a result only fields annihilated by σ_- are not auxiliary. Taking into account the gauge transformation (9.4)

$$\delta\mathcal{C}_n=\widehat{D}_0\varepsilon|_n=\sigma_-\varepsilon_{n+1}+D_0^L\varepsilon_n+\Big(\sigma_+\sum_{m\leqslant n-1}\varepsilon_m\Big)\Big|_n \tag{9.6}$$

one observes that, due to the first term in this transformation law, all components $\mathcal{C}_n$ which are σ_- exact, *i.e.* which belong to the image of σ_-, are Stueckelberg and they can be gauged away. Therefore, a nontrivial dynamical p-form field in $\mathcal{C}$ should belong to the quotient $Ker(\sigma_-)/Im(\sigma_-)$. $\square$

For Einstein-Cartan's gravity, the Stueckelberg gauge symmetry is the local Lorentz symmetry and indeed what distinguishes the frame field from the the metric tensor is that the latter actually belongs to the cohomology $H^1(\sigma_-)$ while the former contains a σ_- exact part.

Proof of B: The proof follows the same lines as the proof of A. The first step has already been performed in the sense that (9.6) already told us that the parameters such that $\sigma_-\varepsilon\neq0$ are Stueckelberg and can be used to completely gauge away trivial parts of the field $\mathcal{C}$. Thus differential parameters must be σ_- closed. The only subtlety is that one should make use of the fact that the gauge transformation $\delta_\varepsilon\mathcal{C}=\widehat{D}_0\varepsilon$ are reducible. More precisely, gauge parameters obeying the reducibility identity

$$\varepsilon=\widehat{D}_0\zeta \tag{9.7}$$

[22]Recall that, at the linearized level, the metric tensor corresponds to the symmetric part $e_{\{\mu\; a\}}$ of the frame field. The antisymmetric part of the frame field $e_{[\mu\; a]}$ can be gauged away by fixing locally the Lorentz symmetry, because it contains as many independent components as the Lorentz gauge parameter ε^{cd}.

[23]Here we use the fact that the operator σ_- acts vertically (that is, it does not differentiate space-time coordinates) thus giving rise to algebraic conditions which express auxiliary fields via derivatives of the other fields.

are trivial in the sense that they do not perform any gauge transformation, $\delta_{\varepsilon=\widehat{D}_0\zeta}\mathcal{C}=0$. The second step of the proof is a mere decomposition of the reducibility identity (9.7) in order to see that σ_- exact parameters correspond to reducible gauge transformations[24]. $\square$

Proof of C: Given a nonnegative integer number n_0, let us suppose that one has already obtained and analyzed (9.1) in grades ranging from $n=0$ up to $n=n_0-1$. Let us analyze (9.1) in grade G equal to n_0 by looking at the constraints imposed by the Bianchi identities. Applying the operator $\widehat{D}_0$ on the covariant derivative $\widehat{D}_0\mathcal{C}$ gives identically zero, which is the Bianchi identity $(\widehat{D}_0)^2\mathcal{C}=0$. Decomposing the latter Bianchi identity gives, in grade equal to n_0-1,

$$(\widehat{D}_0)^2\mathcal{C}|_{n_0-1}=\sigma_-\Big(\widehat{D}_0\mathcal{C}|_{n_0}\Big)+D_0^L\Big(\widehat{D}_0\mathcal{C}|_{n_0-1}\Big)+\Big(\sigma_+\sum_{m\leqslant n_0-2}\widehat{D}_0\mathcal{C}|_m\Big)\Big|_{n_0-1}=0\,. \tag{9.8}$$

By the induction hypothesis, the equations $\widehat{D}_0\mathcal{C}|_m=0$ with $m\leqslant n_0-1$ have already been imposed and analyzed. Therefore (9.8) leads to

$$\sigma_-\Big(\widehat{D}_0\mathcal{C}|_{n_0}\Big)=0\,.$$

In other words, $\widehat{D}_0\mathcal{C}|_{n_0}$ belongs to $Ker(\sigma_-)$. Thus it can contain a σ_- exact part and a nontrivial cohomology part:

$$\widehat{D}_0\mathcal{C}|_{n_0}=\sigma_-(E_{n_0+1})+F_{n_0}\,,\qquad F_{n_0}\in H^{p+1}(\sigma_-)\,.$$

The exact part can be compensated by a field redefinition of the component $\mathcal{C}_{n_0+1}$ which was not treated before (by the induction hypothesis). More precisely, if one performs

$$\mathcal{C}_{n_0+1}\rightarrow\mathcal{C}'_{n_0+1}:=\mathcal{C}_{n_0+1}-E_{n_0+1}\,,$$

then one is left with $\widehat{D}_0\mathcal{C}'|_{n_0}=F_{n_0}$. The field equation (9.1) in grade n_0 is $\widehat{D}_0\mathcal{C}'|_{n_0}=0$. This not only expresses the auxiliary p-forms $\mathcal{C}'_{n_0+1}$ (that are not annihilated by σ_-) in terms of derivatives of lower grade p-forms $\mathcal{C}_k$ ($k\leqslant n_0$), but also sets F_{n_0} to zero. This imposes some $\mathcal{C}_{n_0+1}$-independent conditions on the derivatives of the fields $\mathcal{C}_k$ with $k\leqslant n_0$, thus leading to differential restrictions on the nontrivial dynamical fields. Therefore, to each representative of $H^{p+1}(\sigma_-)$ corresponds a differential field equation. $\square$

Note that if $H^{p+1}(\sigma_-)=0$, the equation (9.1) contains only constraints which express auxilary fields via derivatives of the dynamical fields, imposing no restrictions on the latter. If D_0^L is a first order differential operator and if σ_+ is at most a second order differential operator (which is true in HS applications) then, if $H^{p+1}(\sigma_-)$ is nonzero in the grade k sector, the associated differential equations on a grade ℓ dynamical field are of order $k+1-\ell$. In the next section, two concrete examples of operator σ_- will be considered in many details, together with the physical interpretation of their cohomologies.

10 σ_- cohomology in higher spin gauge theories

As was shown in the previous section, the analysis of generic unfolded dynamical equations amounts to the computation of the cohomology of σ_-. In this section we apply this technique to the analysis of the specific case we focus on: the free unfolded HS gauge field equations of Section 8. The computation of the σ_- cohomology groups relevant for the zero and one-form sectors of the theory is sketched in the subsection 10.1. The physical interpretation of these cohomological results is discussed in the subsection 10.2.

[24]Note that factoring out the σ_- exact parameters accounts for algebraic reducibility of gauge symmetries. The gauge parameters in $H^{p-1}(\sigma_-)$ may still have differential reducibility analogous to differential gauge symmetries for nontrivial dynamical fields. For the examples of HS systems considered below the issue of reducibility of gauge symmetries is irrelevant however because there are no p-form gauge parameters with $p>0$.

10.1 Computation of some σ_- cohomology groups

As explained in Section 8, the fields entering in the unfolded formulation of the HS dynamics are either zero- or one-forms both taking values in various two-row traceless (*i.e.* Lorentz irreducible) Young tableaux. These two sectors of the theory have distinct σ_- operators and thus require separate investigations.

Following Section 3, two-row Young tableaux in the symmetric basis can be conveniently described as a subspace of the polynomial algebra $\mathbb{R}[Y^a, Z^b]$ generated by the $2d$ commuting generators Y^a and Z^b. (One makes contact with the HS algebra convention via the identification of variables (Y, Z) with (Y_1, Y_2). Also let us note that the variable Z^a in this section has no relation with the variables Z_i^A of sections 13 and 14.) Vectors of the space $\Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$ are p-forms taking values in $\mathbb{R}[Y, Z]$. A generic element reads

$$\alpha = \alpha_{a_1 \ldots a_{s-1},\, b_1 \ldots b_t}(x, dx) Y^{a_1} \ldots Y^{a_{s-1}} Z^{b_1} \ldots Z^{b_t}\,,$$

where $\alpha_{a_1 \ldots a_{s-1},\, b_1 \ldots b_t}(x, dx)$ are differential forms. The Lorentz irreducibility conditions of the HS fields are two-fold. Firstly, there is the Young tableau condition (3.1), *i.e.* in this case

$$Y^a \frac{\partial}{\partial Z^a}\, \alpha = 0\,. \tag{10.1}$$

The condition singles out an irreducible $gl(d)$-module $W \subset \Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$. Secondly, the HS fields are furthermore irreducible under $o(d-1, 1)$, which is equivalent to the tracelessness condition (3.6), *i.e.* in our case it is sufficient to impose

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b}\, \alpha = 0\,. \tag{10.2}$$

This condition further restricts to the irreducible $o(d-1, 1)$-module $\hat{W} \subset W$. Note that from (10.1) and (10.2) follows that all traces are zero

$$\eta^{ab} \frac{\partial^2}{\partial Z^a \partial Y^b}\, \alpha = 0\,, \qquad \eta^{ab} \frac{\partial^2}{\partial Z^a \partial Z^b}\, \alpha = 0\,.$$

Connection one-form sector

By looking at the definition (8.1) of the linearized curvatures, and taking into account that the Λ-dependent terms in this formula denote some operator σ_+ that increases the number of Lorentz indices, it should be clear that the σ_- operator of the unfolded field equation (8.2) acts as

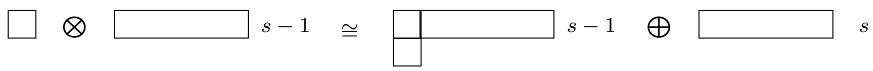
$$\sigma_-\, \omega(Y, Z) \,\propto\, e_0^a \frac{\partial}{\partial Z^a}\, \omega(Y, Z)\,. \tag{10.3}$$

In other words, σ_- is the "de Rham differential" of the "manifold" parametrized by the Z-variables where the generators dZ^a of the exterior algebra are identified with the background vielbein one-forms e_0^a. This remark is very helpful because it already tells us that the cohomology of σ_- is zero in the space $\Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$ with $p > 0$ because its topology is trivial in the Z-variable sector. The actual physical situation is less trivial because one has to take into account the Lorentz irreducibility properties of the HS fields. Both conditions, (10.1) and (10.2), do commute with σ_-, so one can restrict the cohomology to the corresponding subspaces. Since the topology in Z space is not trivial any more, the same is true for the cohomology groups.

As explained in Section 9, for HS equations formulated in terms of the connection one-forms ($p = 1$), the cohomology groups of dynamical relevance are $H^q(\sigma_-)$ with $q = 0, 1$ and 2. The computation of the cohomology groups obviously increases in complexity as the form degree increases. In our analysis we consider simultaneously the cohomology $H^p(\sigma_-, W)$ of traceful two-row Young tableaux (*i.e.* relaxing the tracelessness condition (10.2)) and the cohomology $H^p(\sigma_-, \hat{W})$ of traceless two-row Young tableaux.

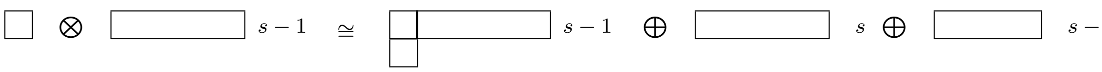
Form degree zero: This case corresponds to the gauge parameters ε. The cocycle condition $\sigma_-\varepsilon = 0$ states that the gauge parameters do not depend on Z. In addition, they cannot be σ_--exact since they are at the bottom of the form degree ladder. Therefore, the elements of $H^0(\sigma_-, W)$ are the completely symmetric tensors which correspond to the unconstrained zero-form gauge parameters $\varepsilon(Y)$ in the traceful case (like in [73]), while they are furthermore traceless in $H^0(\sigma_-, \hat{W})$ and correspond to Fronsdal's gauge parameters [36] $\eta^{ab}\frac{\partial^2}{\partial Y^a \partial Y^b}\varepsilon(Y) = 0$ in the traceless case.

Form degree one: Because of the Poincaré Lemma, any σ_- closed one-form $\alpha(Y, Z)$ admits a representation
$$\alpha(Y, Z) = e_0^a \frac{\partial}{\partial Z^a}\phi(Y, Z)\,. \tag{10.4}$$
The right hand side of this relation should satisfy the Young condition (10.1), *i.e.*, taking into account that it commutes with σ_-,
$$e_0^a \frac{\partial}{\partial Z^a}\Big(Y^b \frac{\partial}{\partial Z^b}\phi(Y, Z)\Big) = 0\,.$$
From here it follows that $\phi(Y, Z)$ is either linear in Z^a (a Z-independent ϕ does not contribute to (10.4)) or satisfies the Young property itself. In the latter case the $\alpha(Y, Z)$ given by (10.4) is σ_- exact. Therefore, nontrivial cohomology can only appear in the sector of elements of the form $\alpha(Y, Z) = e_0^a \beta_a(Y)$, where $\beta_a(Y)$ are arbitrary in the traceful case of W and harmonic in Y in the traceless case of $\hat{W}$. Decomposing $\beta_a(Y)$ into irreps of $gl(d)$



one observes that the hook (*i.e.*, the two-row tableau) is the σ_- exact part, while the one-row part describes $H^1(\sigma_-, W)$. These are the rank s totally symmetric traceful dynamical fields which appear in the unconstrained approach [37, 73].

In the traceless case, decomposing $\beta_a(Y)$ into irreps of $o(d-1, 1)$ one obtains



(10.5)

where all tensors associated with the various Young tableaux are traceless. Again, the hook (*i.e.* two-row tableau) is the σ_- exact part, while the one-row traceless tensors in (10.5) describe $H^1(\sigma_-, \hat{W})$ which just matches the Fronsdal fields [36] because a rank s double traceless symmetric tensor is equivalent to a pair of rank s and rank $s-2$ traceless symmetric tensors.

Form degree two: The analysis of $H^2(\sigma_-, W)$ and $H^2(\sigma_-, \hat{W})$ is still elementary, but a little bit more complicated than that of $H^0(\sigma_-)$ and $H^1(\sigma_-)$. Skipping technical details we therefore give the final results.

By following a reasoning similar to the one in the previous proof, one can show that, in the traceful case, the cohomology group $H^2(\sigma_-, W)$ is spanned by two-forms of the form
$$F \,=\, e_0^a\, e_0^b\, \frac{\partial^2}{\partial Y^a \partial Z^b}\, C(Y, Z)\,, \tag{10.6}$$
where the zero-form $C(Y, Z)$ satisfies the Howe dual $sp(2)$ invariance conditions
$$Z^b \frac{\partial}{\partial Y^b} C(Y, Z) = 0\,, \qquad Y^b \frac{\partial}{\partial Z^b} C(Y, Z) = 0\,, \tag{10.7}$$

and, therefore,

$$(Z^b \frac{\partial}{\partial Z^b} - Y^b \frac{\partial}{\partial Y^b}) C(Y,Z) = 0\,. \tag{10.8}$$

In accordance with the analysis of Section 3, this means that

$$C(Y,Z) = C_{a_1\dots a_s,\, b_1\dots b_s} Y^{a_1} \dots Y^{a_s} Z^{b_1} \dots Z^{b_s}\,, \tag{10.9}$$

where the zero-form components $C_{a_1\dots a_s,\, b_1\dots b_s}$ have the symmetry properties corresponding to the rectangular two-row Young tableau of length s

$$C_{a_1\dots a_s,\, b_1\dots b_s} \sim \begin{array}{|c|}\hline \qquad\qquad\qquad \\ \hline \qquad\qquad\qquad \\ \hline \end{array}\begin{array}{l} s \\ s \end{array}\quad . \tag{10.10}$$

From (10.6) it is clear that F is σ_- closed and $F \in W$. It is also clear that it is not σ_- exact in the space W. Indeed, suppose that $F = \sigma_- G$, $G \in W$. For any polynomial $G \in W$ its power in Z cannot be higher than the power in Y (because of the Young property, the second row of a Young tableau is not longer than the first row). Since σ_- decreases the power in Z, the degree in Z of exact elements $\sigma_- G$ is strictly less than the degree in Y. This is not true for the elements (10.6) because of the condition (10.8). The tensors (10.9) correspond to the linearized curvature tensors introduced by de Wit and Freedman [37].

Let us now consider the traceless case of $H^2(\sigma_-, \hat{W})$. The formula (10.6) still gives cohomology but now $C(Y,Z)$ must be traceless,

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} C(Y,Z) = 0\,, \qquad \eta^{ab} \frac{\partial^2}{\partial Z^a \partial Z^b} C(Y,Z) = 0\,, \qquad \eta^{ab} \frac{\partial^2}{\partial Y^a \partial Z^b} C(Y,Z) = 0\,.$$

In this case the tensors (10.9) correspond to Weyl-like tensors, *i.e.* on-shell curvatures. They form the so-called "Weyl cohomology". But this is not the end of the story because there are other elements in $H^2(\sigma_-, \hat{W})$. They span the "Einstein cohomology" and contain two different types of elements:

$$r_1 = e_0^a e_0^b \Big((Z_b\, Y^c - Y_b\, Z^c) \frac{\partial^2}{\partial Y^a \partial Y^c} \rho_1(Y) \Big)\,, \tag{10.11}$$

$$\begin{aligned} r_2 \;=\;& e_0^a e_0^b \Big((d-1)(d + Y^c \frac{\partial}{\partial Y^c} - 2) Y_a Z_b + d Y_c Y^c Z_a \frac{\partial}{\partial Y^b} \\ -\;& (d + Y^c \frac{\partial}{\partial Y^c} - 2) Y_e Z^e Y_a \frac{\partial}{\partial Y^b} + Y_e Y^e Y_a Z^c \frac{\partial}{\partial Y^c} \frac{\partial}{\partial Y^b} \Big) \rho_2(Y)\,, \end{aligned} \tag{10.12}$$

where $\rho_{1,2}(Y)$ are arbitrary harmonic polynomials

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} \rho_{1,2}(Y) = 0\ ,$$

thus describing completely symmetric traceless tensors.

One can directly see that r_1 and r_2 belong to $\hat{W}$ (*i.e.*, satisfy (10.1) and (10.2)) and are σ_- closed, $\sigma_- r_{1,2} = 0$ (the check is particularly simple for r_1). It is also easy to see that $r_{1,2}$ are in the nontrivial cohomology class. Indeed, the appropriate trivial class is described in tensor notations by the two-form $e_0^c \omega_{a_1\dots a_{s-1},bc}$, where

$$\omega_{a_1\dots a_{s-1},bc} = e_0^f \omega_{f;\, a_1\dots a_{s-1},bc} \tag{10.13}$$

is a one-form that has the properties of traceless two-row Young tableau with $s-1$ cells in the first row and two cells in the second row. The trivial cohomology class neither contains a rank $s-2$ tensor like ρ_2, that needs a double contraction in $\omega_{f;a_1\dots a_{s-1},bc}$ in (10.13), nor a rank s symmetric tensor like ρ_1, because symmetrization of a contraction of the tensor $\omega_{f;\, a_1\dots a_{s-1},bc}$ over any s indices gives zero. It can be shown that the Einstein cohomology (10.11) and (10.12) together with the Weyl cohomology (10.6) span $H^2(\sigma_-, \hat{W})$.

Weyl zero-form sector

The Λ-dependent terms in the formula (8.3) denote some operator σ_+ that increases the number of Lorentz indices, therefore the operator σ_- of the unfolded field equation (8.3) is given by

$$(\sigma_- C)^{a_1\dots a_{s+k},\, b_1\dots b_s} = -e_{0\,c}\Big((2+k)C^{a_1\dots a_{s+k}c,\, b_1\dots b_s} + s\, C^{a_1\dots a_{s+k}\{b_1,\, b_2\dots b_s\}c}\Big) \tag{10.14}$$

both in the traceless and in the traceful twisted adjoint representations. One can analogously compute the cohomology groups $H^q(\sigma_-, W)$ and $H^q(\sigma_-, \hat{W})$ with $q = 0$ and 1, which are dynamically relevant in the zero-from sector ($p = 0$).

Form degree zero: It is not hard to see that $H^0(\sigma_-, W)$ and $H^0(\sigma_-, \hat{W})$ consist, respectively, of the generalized Riemann (*i.e.* traceful) and Weyl (*i.e.* traceless) tensors $C^{a_1\dots a_s,\, b_1\dots b_s}$.

Form degree one: The cohomology groups $H^1(\sigma_-, W)$ and $H^1(\sigma_-, \hat{W})$ consist of one-forms of the form

$$w^{a_1\dots a_s, b_1,\dots b_s} = e_{0\,c} C^{a_1\dots a_s\,, b_1\dots b_s\,, c}\,, \tag{10.15}$$

where $C^{a_1\dots a_s\,, b_1\dots b_s\,, c}$ is, respectively, a traceful and traceless zero-form described by the three row Young tableaux with s cells in the first and second rows and one cell in the third row. Indeed, a one-form $w^{a_1\dots a_s, b_1,\dots b_s}$ given by (10.15) is obviously σ_- closed (by definition, σ_- gives zero when applied to a rectangular Young tableau) and not σ_- exact, thus belonging to nontrivial cohomology.

In addition, in the traceless case, $H^1(\sigma_-, \hat{W})$ includes the "Klein-Gordon cohomology"

$$w^a = e_0^a k \tag{10.16}$$

and the "Maxwell cohomology"

$$w^{a,\, b} = e_0^a m^b - e_0^b m^a\,, \tag{10.17}$$

where k and m^a are arbitrary scalar and vector, respectively. It is obvious that the one-forms (10.16) and (10.17) are σ_- closed for σ_- (10.14). They are in fact exact in the traceful case with $w = \sigma_-(\varphi)$, where

$$\varphi^{a_1 a_2} \propto k\eta^{a_1 a_2}\,, \qquad \varphi^{a_1 a_2\,, b} \propto 2m^b\eta^{a_1 a_2} - m^{a_2}\eta^{a_1 b} - m^{a_1}\eta^{a_2 b} \tag{10.18}$$

but are not exact in the traceless case since the tensors φ in (10.18) are not traceless although the resulting $w = \sigma_-(\varphi)$ belong to the space of traceless tensors.

Unfolded system

One may combine the connection one-forms ω and Weyl zero-forms C into the set $\mathcal{C} = (\omega, C)$ and redefine $\sigma_- \to \hat{\sigma}_-$ in such a way that $\hat{\sigma}_- = \sigma_- + \Delta\sigma_-$ where σ_- acts on ω and C following (10.3) and (10.14), respectively, and $\Delta\sigma_-$ maps the rectangular zero-form Weyl tensors to the two-form sector via

$$\Delta\sigma_-\, C(Y,Z) \,\propto\, e_0^a\, e_0^b\, \frac{\partial^2}{\partial Y^a \partial Z^b}\, C(Y,Z)\,, \tag{10.19}$$

thus adding the term with the Weyl zero-form to the linearized curvature of one-forms.

The dynamical content of the unfolded system of equations (8.2)-(8.3) is encoded in the cohomology groups of $\hat{\sigma}_-$. The gauge parameters are those of $H^0(\hat{\sigma}_-)$ in the adjoint module (*i.e.* the connection one-form sector). The dynamical fields are symmetric tensors of $H^1(\hat{\sigma}_-)$ in the adjoint module, along with the scalar field in the zero-form sector. There are no nontrivial field equations in the traceful case. As explained in more detail in Subsection 10.2, in the traceless case, the field equations are associated with the Einstein cohomology (10.11)-(10.12), Maxwell cohomology (10.17) and Klein-Gordon cohomology (10.16). Note that the cohomology (10.15) disappears as a result of gluing the one-form adjoint and zero-form twisted adjoint modules by (10.19).

10.2 Physical interpretation of some σ_- cohomology groups

These cohomological results tell us that there are several possible choices for gauge invariant differential equations on HS fields.

The form of $r_{1,2}$ (10.11) and (10.12) indicates that the Einstein cohomology is responsible for the Lagrangian field equations of completely symmetric double traceless fields. Indeed, carrying one power of Z^a, they are parts of the HS curvatures $R_{a_1\ldots a_{s-1},b}$ with one cell in the second row of the corresponding Young tableau. For spin s, $\rho_1(Y)$ is a harmonic polynomial of homogeneity degree s, while $\rho_2(Y)$ is a harmonic polynomial of homogeneity degree $s-2$. As a result, the field equations which follow from $r_1 = 0$ and $r_2 = 0$ are of second order[25] in derivatives of the dynamical Fronsdal fields taking values in $H^1(\sigma_-, \hat{W})$ and, as expected for Lagrangian equations in general, there are as many equations as dynamical fields.

For example, spin-2 equations on the trace and traceless parts of the metric tensor associated with the elements of $H^1(\sigma_-, \hat{W})$ result from the conditions $r_1 = 0$ and $r_2 = 0$ with $\rho_{1,2}$ in (10.11) and (10.12) of the form $\rho_1(Y) = r_{ab}Y^aY^b$, $\rho_2(Y) = r$, with arbitrary r and traceless r_{ab}. These are, respectively, the traceless and trace parts of the linearized Einstein equations. Analogously, the equations $r_1 = 0$ and $r_2 = 0$ of higher orders in Y correspond to the traceless and first trace parts of the Fronsdal spin $s > 1$ HS equations (which are, of course, double traceless).

Thus, in the traceless case, the proper choice to reproduce dynamical field equations equivalent to the equations resulting from the Fronsdal Lagrangian is to keep only the Weyl cohomology nonzero. Setting elements of the Einstein cohomology to zero, which imposes the second-order field equations on the dynamical fields, leads to

$$R_1 \;=\; e_0^a\, e_0^b\, \epsilon_{ij}\, \frac{\partial^2}{\partial Y_i^a \partial Y_j^b}\, C(Y_k^c)\,, \tag{10.20}$$

where one makes contact with the HS algebra convention via the identification of variables (Y, Z) with (Y_1, Y_2). Here $C(Y_k^c)$ satisfies the $sp(2)$ invariance condition

$$Y_i^a \frac{\partial}{\partial Y_j^a} C(Y) = \frac{1}{2}\delta_j^i Y_k^a \frac{\partial}{\partial Y_k^a} C(Y) \tag{10.21}$$

along with the tracelessness condition

$$\eta^{ab} \frac{\partial^2}{\partial Y_j^a \partial Y_i^b} C(Y) = 0 \tag{10.22}$$

to parametrize the Weyl cohomology. The equation (10.20) is exactly (8.2). Thus, the generalized Weyl tensors on the right hand side of (8.2) parametrize the Weyl cohomology in the HS curvatures exactly so as to make the equations (8.2) for $s > 1$ equivalent to the HS field equations that follow from Fronsdal's action.[26]

Alternatively, one can set the Weyl cohomology to zero, keeping the Einstein cohomology arbitrary. It is well known that in the spin-2 case of gravity the generic solution of the condition that the Weyl tensor is zero leads to conformally flat metrics. Analogous analysis for free spin 3 was performed in [74]. It is tempting to conjecture that this property is true for any spin $s > 2$, *i.e.,* the condition that Weyl cohomology is zero singles out the "conformally flat" single trace HS fields of the form

$$\varphi_{\nu_1\ldots\nu_s}(x) = g_{\{\nu_1\nu_2}(x)\psi_{\nu_3\ldots\nu_s\}}(x) \tag{10.23}$$

with traceless symmetric $\psi_{\nu_3\ldots\nu_s}(x)$. Indeed, the conformally flat free HS fields (10.23) have zero generalized Weyl tensor simply because it is impossible to build a traceless tensor (10.10) from derivatives of

[25] The cohomological analysis outlined here can be extended to the space $\hat{W}_n$ of tensors required to have their n-th trace equal zero, *i.e.* with (10.2) replaced by $(\eta^{ab}\frac{\partial^2}{\partial Y^a \partial Y^b})^n \alpha(Y, Z) = 0$. It is tempting to conjecture that the resulting gauge invariant field equations will contain $2n$ derivatives.

[26] The subtle relationship between Fronsdal's field equations and the tracelessness of the HS curvatures was discussed in details for metric-like spin-3 fields in [74].

$\psi_{\nu_1 \ldots \nu_{s-2}}(x)$.

The spin-0 and spin-1 equations are not described by the cohomology in the one-form adjoint sector. The Klein-Gordon and Maxwell equations result from the cohomology of the zero-form twisted adjoint representation, which encodes equations that can be imposed in terms of the generalized Weyl tensors which contain the spin-1 Maxwell tensor and the spin-0 scalar as the lower spin particular cases. More precisely, the spin-0 and spin-1 field equations are contained in the parts of equations (8.3) associated with the cohomology groups (10.16) and (10.17). For spin $s \geqslant 1$, the cohomology (10.15) encodes the Bianchi identities for the definition of the Weyl (Riemann) tensors by (10.20). This is equivalent to the fact that, in the traceless case, the equations (10.20) and (8.3) describe properly free massless equations of all spins supplemented with an infinite set of constraints for auxiliary fields, which is the content of the central on-mass-shell theorem.

In the traceful case, the equation (10.20) with traceful $C(Y^a_i)$ does not impose any differential restrictions on the fields in $H^1(\sigma_-, W)$ because $e^a_0\, e^b_0\, \epsilon_{ij}\, \frac{\partial^2}{\partial Y^a_i \partial Y^b_j}\, C(Y^c_k)$ span the full $H^2(\sigma_-, W)$ of the one-form adjoint sector. This means that the equation (10.20) describes a set of constraints which express all fields in terms of derivatives of the dynamical fields in $H^1(\sigma_-, W)$. In this sense, the equation (10.20) for a traceful field describes off-mass-shell constraints identifying the components of $C(Y)$ with the deWit-Freedman curvature tensors[27]. Supplementing (10.20) by the covariant constancy equation (8.3) on the traceful zero-forms $C(Y)$ one obtains an infinite set of constraints for any spin which express infinite sets of auxiliary fields in terms of derivatives of the dynamical fields, imposing no differential restrictions on the latter. These constraints provide unfolded off-mass-shell description of massless fields of all spins. We call this fact "central off-mass-shell theorem".

Let us stress that our analysis works both in the flat space-time and in the $(A)dS_d$ case originally considered in [41]. Indeed, although the nonzero curvature affects the explicit form of the background frame and the Lorentz covariant derivative D^L_0 and also requires a nonzero operator σ_+ denoted by $O(\Lambda)$ in (8.1), all this does not affect the analysis of the σ_- cohomology because the operator σ_- remains of the form (10.3) with a nondegenerate frame field e^a_μ.

Let us make the following comment. The analysis of the dynamical content of the covariant constancy equations $\widehat{D}_0 \mathcal{C} = 0$ may depend on the choice of the grading operator G and related graded decomposition (9.2). This may lead to different definitions of σ_- and, therefore, different interpretations of the same system of equations. For example, one can choose a different definition of σ_- in the space W of traceful tensors simply by decomposing W into a sum of irreducible Lorentz tensors (*i.e.*, traceless tensors) and then defining σ_- within any of these subspaces as in $\hat{W}$. In this basis, the equations (8.2) will be interpreted as dynamical equations for an infinite set of traceless dynamical fields. This phenomenon is not so surprising, taking into account the well-known analogous fact that, say, an off-mass-shell scalar can be represented as an integral over the parameter of mass of an infinite set of on-mass-shell scalar fields. More generally, to avoid paradoxical conclusions one has to take into account that σ_- may or may not have a meaning in terms of the elements of the Lie algebra Lie h that gives rise to the covariant derivative (9.2).

10.3 Towards nonlinear equations

Nonlinear equations should replace the linearized covariant derivative $\widetilde{D}_0$ with the full one, $\widetilde{D}$, containing the h-valued connection ω. They should also promote the linearized curvature R_1 to R. Indeed, (8.3) and (10.20) cannot be correct at the nonlinear level because the consistency of $\widetilde{D}C = 0$ implies arbitrarily high powers of C in the r.h.s. of the modified equations, since

$$\widetilde{D}\widetilde{D}C \sim RC \sim O(C^2) + \text{higher order terms}\ ,$$

[27]However, as pointed out in [75], if one imposes $C(Y^a_i)$ to be harmonic in Y, then the corresponding field equations (10.20) imposes the deWit-Freedman curvature to be traceless. In this sense, it may be possible to remove the tracelessness requirement in the frame-like formulation (see Section 4.2) without changing the physical content of the free field equations.

the last relation being motivated by (10.20).

Apart from dynamical field equations, the unfolded HS field equations contain constraints on the auxiliary components of the HS connections, expressing the latter via derivatives of the nontrivial dynamical variables (*i.e.* Fronsdal fields), modulo pure gauge ambiguity. Originally, all HS gauge connections ${\omega_\mu}^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ have dimension $length^{-1}$ so that the HS field strength (5.15) needs no dimensionful parameter to have dimension $length^{-2}$. However, this means that when some of the gauge connections are expressed via derivatives of the others, these expressions must involve space-time derivatives in the dimensionless combination $\rho\frac{\partial}{\partial x^\nu}$, where ρ is some parameter of dimension *length*. The only dimensionful parameter available in the analysis of the free dynamics is the radius ρ of the AdS space-time related to the cosmological constant by (2.8). Recall that it appears through the definition of the frame field (2.10) with $V^A \sim \rho$ adapted to make the frame E^A_μ (and, therefore, the metric tensor) dimensionless. As a result, the HS gauge connections are expressed by the unfolded field equations through the derivatives of the dynamical fields as

$$\omega^{\;a_1\ldots a_{s-1},\,b_1\ldots b_t\hat{d}\ldots\hat{d}} = \Pi\left(\rho^t\frac{\partial}{\partial x^{b_1}}\cdots\frac{\partial}{\partial x^{b_t}}\omega^{\;a_1\ldots a_{s-1},\,\hat{d}\ldots\hat{d}}\right) + \text{lower derivative terms}\,, \qquad (10.24)$$

where Π is some projector that permutes indices (including the indices of the forms) and projects out traces. Plugging these expressions back into the HS field strength (5.15) one finds that HS connections with $t > 1$ (*i.e.* extra fields that appear for $s > 2$) contribute to the terms with higher derivatives which blow up in the flat limit $\rho \to \infty$. This mechanism brings higher derivatives and negative powers of the cosmological constant into HS interactions (but not into the free field dynamics because the free action is required to be independent of the extra fields). Note that a similar phenomenon takes place in the sector of the generalized Weyl zero-forms $C(Y)$ in the twisted adjoint representation.

11 Star product

We shall formulate consistent nonlinear equations using the star product. In other words we shall deal with ordinary commuting variables Y^A_i instead of operators $\hat{Y}^A_i$. In order to avoid ordering ambiguities, we use the Weyl prescription. An operator is said to be Weyl ordered if it is completely symmetric under the exchange of operators $\hat{Y}^A_i$. One establishes a one to one correspondence between each Weyl ordered polynomial $f(\hat{Y})$ (5.4) and its symbol $f(Y)$, defined by substituting each operator $\hat{Y}^A_i$ with the commuting variable Y^A_i. Thus $f(Y)$ admits a formal expansion in power series of Y^A_i identical to that of $f(\hat{Y})$, *i.e.* with the same coefficients,

$$f(Y) = \sum_{m,n} f_{A_1\ldots A_m\,,B_1\ldots B_n} Y_1^{A_1}\ldots Y_1^{A_m} Y_2^{B_1}\ldots Y_2^{B_n}\,. \qquad (11.1)$$

To reproduce the algebra A_{d+1}, one defines the *star product* in such a way that, given any couple of functions f_1, f_2, which are symbols of operators $\hat{f}_1, \hat{f}_2$ respectively, $f_1 * f_2$ is the symbol of the operator $\hat{f}_1\hat{f}_2$. The result is nontrivial because the operator $\hat{f}_1\hat{f}_2$ should be Weyl ordered. It can be shown that this leads to the well-known Weyl-Moyal formula

$$(f_1 * f_2)(Y) = f_1(Y)\, e^{\frac{1}{2}\overleftarrow{\partial}^j_A \overrightarrow{\partial}^i_B \eta^{AB}\epsilon_{ji}}\, f_2(Y)\;, \qquad (11.2)$$

where $\partial^j_A \equiv \frac{\partial}{\partial Y^A_j}$ and $\overleftarrow{\partial}$, as usual, means that the partial derivative acts to the left while $\overrightarrow{\partial}$ acts to the right. The rationale behind this definition is simply that higher and higher powers of the differential operator in the exponent produce more and more contractions. One can show that the star product is an associative product law, and that it is regular, which means that the star product of two polynomials in Y is still a polynomial. From (11.2) it follows that the star product reproduces the proper commutation relation of oscillators,

$$[Y^A_i, Y^B_j]_* \equiv Y^A_i * Y^B_j - Y^B_j * Y^A_i = \epsilon_{ij}\eta^{AB}\;.$$

The star product has also an integral definition, equivalent to the differential one given by (11.2), which is

$$(f_1 * f_2)(Y) = \frac{1}{\pi^{2(d+1)}} \int dSdT\, f_1(Y+S)\, f_2(Y+T) \exp(-2S_i^A T_A^i) \ . \tag{11.3}$$

The whole discussion of Section 5 can be repeated here, with the prescription of substituting operators with their symbols and operator products with star products. For example, the $o(d-1,2)$ generators (5.5) and the $sp(2)$ generators (5.6) are realized as

$$T^{AB} = -T^{BA} = \frac{1}{2} Y^{iA} Y_i^B \ , \quad t_{ij} = t_{ji} = Y_i^A Y_{jA} \ , \tag{11.4}$$

respectively. Note that

$$Y_i^A * = Y_i^A + \frac{1}{2} \frac{\overrightarrow{\partial}}{\partial Y_A^i}$$

and

$$*Y_i^A = Y_i^A - \frac{1}{2} \frac{\overleftarrow{\partial}}{\partial Y_A^i} \ .$$

From here it follows that

$$[Y_i^A \, , f(Y)]_* = \frac{\partial}{\partial Y_A^i} f(Y) \tag{11.5}$$

and

$$\{Y_i^A \, , f(Y)\}_* = 2 Y_i^A f(Y) \, . \tag{11.6}$$

With the help of these relations it is easy to see that the $sp(2)$ invariance condition $[t_{ij}, f(Y)]_* = 0$ indeed has the form (5.8) and singles out two-row rectangular Young tableaux, *i.e.* it implies that the coefficients $f_{A_1 \ldots A_m \, , B_1 \ldots B_n}$ are nonzero only if $n = m$, and symmetrization of any $m+1$ indices of $f_{A_1 \ldots A_m \, , B_1 \ldots B_m}$ gives zero. Let us also note that if $[t_{ij}, f(Y)]_* = 0$ then

$$t_{ij} * f = f * t_{ij} = \Big(t_{ij} + \frac{1}{4} \frac{\partial^2}{\partial Y_A^i \partial Y^{jA}} \Big) f \, . \tag{11.7}$$

One can then introduce the gauge fields taking values in this star algebra as functions $\omega(Y|x)$ of oscillators,

$$\omega(Y|x) = \sum_{s \geq 1} i^{s-2} \omega^{A_1 \ldots A_{s-1} \, , B_1 \ldots B_{s-1}}(x) Y_{1A_1} \ldots Y_{1A_{s-1}} Y_{2B_1} \ldots Y_{2B_{s-1}} \ , \tag{11.8}$$

with their field strength defined by

$$R(Y) = d\omega(Y) + (\omega * \omega)(Y) \tag{11.9}$$

and gauge transformations

$$\delta\omega(Y) = d\epsilon(Y) + [\omega \, , \epsilon]_*(Y) \tag{11.10}$$

(where the dependence on the space-time coordinates x is implicit). For the subalgebra of $sp(2)$ singlets we have

$$D(t_{ij}) = 0 \, , \qquad [t_{ij} \, , \epsilon]_* = 0 \, , \qquad [t_{ij} \, , R]_* = 0 \, . \tag{11.11}$$

Note that $d(t_{ij}) = 0$ and, therefore from the first of these relations it follows that $[t_{ij} \, , \omega]_* = 0$, which is the $sp(2)$ invariance relation.

Furthermore, one can get rid of traces by factoring out the ideal $\mathcal{I}$ spanned by the elements of the form $t_{ij} * g^{ij}$. For factoring out the ideal $\mathcal{I}$ it is convenient to consider [57] elements of the form $\Delta * g$ where Δ is an element satisfying the conditions $\Delta * t_{ij} = t_{ij} * \Delta = 0$. The explicit form of Δ is [57,75]

$$\Delta = \int_{-1}^{1} ds (1-s^2)^{\frac{1}{2}(d-3)} \exp\left(s\sqrt{z}\,\right) = 2 \int_{0}^{1} ds (1-s^2)^{\frac{1}{2}(d-3)} ch(s\sqrt{z}\,) \, , \tag{11.12}$$

where

$$z = \frac{1}{4} Y_i^A Y_{Aj} Y^{Bi} Y_B^j \,. \tag{11.13}$$

Indeed, one can see [57] that $\Delta * f = f * \Delta$ and that all elements of the form $f = u^{ij} * t_{ij}$ or $f = t_{ij} * u^{ij}$ disappear in $\Delta * f$, *i.e.* the factorization of $\mathcal{I}$ is automatic. The operator Δ, which we call quasiprojector, is not a projector because $\Delta * \Delta$ does not exist (diverges) [57,75]. One therefore cannot define a product of two elements $\Delta * f$ and $\Delta * g$ in the quotient algebra as the usual star product. A consistent definition for the appropriately modified product law $\circ$ is

$$(\Delta * f) \circ (\Delta * g) = \Delta * f * g \tag{11.14}$$

(for more detail see section 3 of [57]). Note that from this consideration it follows that the star product $g_1 * g_2$ of any two elements satisfying the strong $sp(2)$ invariance condition $t_{ij} * g_{1,2} = 0$ of [75] is ill-defined because such elements admit a form $g_{1,2} = \Delta * g'_{1,2}$.

12 Twisted adjoint representation

As announced in Section 8, we give here a precise definition of the module in which the Weyl-like zero-forms take values, in such a way that (8.3) is reproduced at the linearized level. Taking the quotient by the ideal $\mathcal{I}$ is a subtle step the procedure of which is explained in Subsection 12.2.

12.1 Definition of the twisted adjoint module

To warm up, let us start with the adjoint representation. Let $\mathcal{A}$ be an associative algebra endowed with a product denoted by $*$. The $*$-commutator is defined as $[a,b]_* = a*b - b*a \ , \ a,b \in \mathcal{A}$. As usual for an associative algebra, one constructs a Lie algebra g from $\mathcal{A}$, the Lie bracket of which is the $*$-commutator. Then g has an adjoint representation the module of which coincides with the algebra itself and such that the action of an element $a \in g$ is given by

$$[a, X]_* \ , \forall X \in \mathcal{A} \ .$$

Let τ be an automorphism of the algebra $\mathcal{A}$, that is to say

$$\tau(a*b) = \tau(a) * \tau(b) \,, \qquad \tau(\lambda a + \mu b) = \lambda \tau(a) + \mu \tau(b), \qquad \forall a,b \in \mathcal{A} \,,$$

where λ and μ are any elements of the ground field $\mathbb{R}$ or $\mathbb{C}$. The τ-twisted adjoint representation of g has the same definition as the adjoint representation, except that the action of g on its elements is modified by the automorphism τ:

$$a(X) \ \rightarrow \ a * X - X * \tau(a) \ .$$

It is easy to see that this gives a representation of g.

The appropriate choice of τ, giving rise to the infinite bunch of fields contained in the zero-form C (matter fields, generalized Weyl tensors and their derivatives), is the following:

$$\tau(f(Y)) = \widetilde{f}(Y) \ , \tag{12.1}$$

where

$$\widetilde{f}(Y) = f(\widetilde{Y}) \,, \qquad \widetilde{Y}_i^A = Y_i^A - 2V^A V^B Y_{Bi} \,, \tag{12.2}$$

i.e., $\widetilde{Y}_i^A$ is the oscillator Y_i^A reflected with respect to the compensator (recall that we use the normalization $V_A V^A = 1$). So one can say that the automorphism τ is some sort of parity transformation in the V-direction, leaving unaltered the Lorentz components of the oscillators. More explicitly, in terms

of the transverse and longitudinal components

$$^{\perp}Y_i^A = Y_i^A - V^A V_B Y_i^B\,, \qquad {}^{\|}Y_i^A = V^A V^B Y_{Bi}\,,$$

the automorphism τ is the transformation

$$^{\perp}Y_i^A \to {}^{\perp}Y_i^A\,, \qquad {}^{\|}Y_i^A \to -\,{}^{\|}Y_i^A\,,$$

or, in the standard gauge, $Y_i^a \to Y_i^a$, $Y_i^{\hat d} \to -Y_i^{\hat d}$. From (11.2) it is obvious that τ is indeed an automorphism of the star product algebra.

The automorphism τ (12.1) leaves invariant the $sp(2)$ generators

$$\tau(t_{ij}) = t_{ij}\,.$$

This allows us to require the zero-form $C(Y|x)$ in the twisted adjoint module to satisfy the $sp(2)$ invariance condition

$$t_{ij} * C = C * t_{ij} \tag{12.3}$$

and to define the covariant derivative in the twisted adjoint module of $hu(1|2{:}[d-1,2])$ as

$$\widetilde{D}C = dC + \omega * C - C * \widetilde{\omega}\,. \tag{12.4}$$

At the linearized level one obtains

$$\widetilde{D}_0 C = dC + \omega_0 * C - C * \widetilde{\omega}_0\,. \tag{12.5}$$

One decomposes ω_0 into its Lorentz and translational part, $\omega_0 = \omega_0^L + \omega_0^{transl}$, via (2.15), which gives

$$\begin{aligned}
\omega_0^L &\equiv \frac{1}{2}\,\omega_0^{AB}\,{}^{\perp}Y_A^i\,{}^{\perp}Y_{Bi} = \frac{1}{2}\,\omega_0^{ab}\,Y_a^i\,Y_{bi}\,,\\
\omega_0^{transl} &\equiv \omega_0^{AB}\,{}^{\perp}Y_A^i\,{}^{\|}Y_{Bi} = e_0^a Y_a^i Y_{Bi} V^B\,.
\end{aligned}$$

Taking into account the definition (12.1), it is clear that τ changes the sign of ω_0^{transl} while leaving ω_0^L untouched. This is tantamount to say that $\widetilde{D}_0$ contains an anticommutator with the translational part of the connection instead of a commutator,

$$\widetilde{D}_0 C = D_0^L C + \{\omega_0^{transl}, C\}_*\,,$$

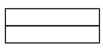
where D_0^L is the usual Lorentz covariant derivative, acting on Lorentz indices. Expanding the star products, we have

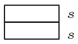
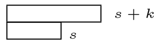
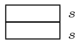
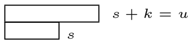
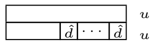
$$\widetilde{D}_0 = D_0^L + 2E_0^A V^B\Big({}^{\perp}Y_A^i\,{}^{\|}Y_{Bi} - \frac{1}{4}\epsilon^{ij}\frac{\partial^2}{\partial\,{}^{\perp}Y^{Ai}\partial\,{}^{\|}Y^{Bj}}\Big)\,, \tag{12.6}$$

the last term being due to the noncommutative structure of the star algebra.

The equation (12.6) suggests that there exists a grading operator

$$N^{tw} = N_\perp - N_\| = {}^{\perp}Y_i^A\frac{\partial}{\partial^{\perp}Y_i^A} - {}^{\|}Y_i^A\frac{\partial}{\partial^{\|}Y_i^A} \tag{12.7}$$

commuting with $\widetilde{D}_0$, and whose eigenvalues $N_\perp - N_\| = 2s$, where s is the spin, classify the various irreducible submodules into which the twisted adjoint module decomposes as $o(d-1,2)$-module. In other words, the system of equations $\widetilde{D}_0 C = 0$ decomposes into an infinite number of independent subsystems, the fields of each subset satisfying $N^{tw}C = 2sC$, for some nonnegative integer s. Let us give some more detail about this fact. Recall that requiring $sp(2)$ invariance restricts us to the rectangular two row AdS_d Young tableaux  . By means of the compensator V^A we then distinguish between transverse (Lorentz) and longitudinal indices. Clearly $N^{tw} \geqslant 0$, since having more than half of vector indices in the

extra direction V would imply symmetrization over more than half of all indices, thus giving zero because of the symmetry properties of Young tableaux. Then, each independent sector $N^{tw} = 2s$ of the twisted adjoint module starts from the rectangular Lorentz-Young tableau corresponding to the (generalized) Weyl tensor  , and admits as further components all its "descendants" , which the equations themselves set equal to k Lorentz covariant derivatives of . From the AdS_d-Young tableaux point of view, the set of fields forming an irreducible submodule of the twisted adjoint module with some fixed s is nothing but the components of the fields $C^{A_1\dots A_u, B_1\dots B_u}$ $(u = s, \dots \infty)$ that have $k = u - s$ indices parallel to V^A:  $\sim$ .

12.2 Factorization procedure

The twisted adjoint module as defined in the previous section is off-mass-shell because the zero-form $C(Y|x)$ is traceful in the oscillators Y_i^A. To put it on-mass-shell one has to factor out those elements of the twisted adjoint module T that also belong to the ideal $\mathcal{I}$ of the associative algebra $\mathcal{S}$ of $sp(2)$ singlets (see Subsection 5.2) and form a submodule $T \cap \mathcal{I}$ of the HS algebra. By a slight abuse of terminology, we will also refer to this submodule as "ideal $\mathcal{I}$" in the sequel. Therefore, "quotienting by the ideal $\mathcal{I}$" means dropping terms $g \in T \cap \mathcal{I}$, that is

$$g \in \mathcal{I} \quad \Longleftrightarrow \quad [t_{ij}, g]_* = 0\,, \quad g = t_{ij} * g^{ij} = g^{ij} * t_{ij}\,. \tag{12.8}$$

The factorization of $\mathcal{I}$ admits an infinite number of possible ways of choosing representatives of the equivalence classes (recall that f_1 and f_2 belong to the same equivalence class iff $f_1 - f_2 \in \mathcal{I}$). The tracelessness condition (3.6), which in the case of HS gauge one-forms amounts to

$$\frac{\partial^2}{\partial Y_i^A \partial Y_{Aj}} \omega(Y) = 0\ ,$$

is not convenient for the twisted adjoint representation because it does not preserve the grading (12.7), *i.e.* it does not commute with N^{tw}. A version of the factorization condition that commutes with N^{tw} and is more appropriate for the twisted adjoint case is

$$\Big(\frac{\partial^2}{\partial^\perp Y_i^A \partial^\perp Y_{Aj}} - 4\,{}^\parallel Y^{Ai\,\parallel} Y_A^j\Big) C(Y) = 0\ .$$

For the computations, both in the adjoint and the twisted adjoint representation, it may be convenient to require Lorentz tracelessness, *i.e.* the tracelessness with respect to transversal indices. Recall that the final result is insensitive to a particular choice of the factorization condition, that is, choosing one or another condition is a matter of convenience. In practice the factorization procedure is implemented as follows. Let A^{tr} denote either a one-form HS connection or a HS Weyl zero-form satisfying a chosen tracelessness (*i.e.* factorization) condition. The left hand side of the field equations contains the covariant derivative D in the adjoint, $D = D_0$, or twisted adjoint, $D = \widetilde{D}_0$, representation. $D(A^{tr})$ does not necessarily satisfy the chosen tracelessness condition, but it can be represented in the form

$$D(A^{tr}) = (D(A^{tr}))^{tr} + t_{ij} X_1^{ij}\,, \tag{12.9}$$

where the ordinary product of t_{ij} with some X_1^{ij} parametrizes the traceful part, while $(D(A^{tr}))^{tr}$ satisfies the chosen tracelessness condition. Note that X_1^{ij} contains less traces than $D(A^{tr})$. One rewrites (12.9) in the form

$$D(A^{tr}) = (D(A^{tr}))^{tr} + t_{ij} * X_1^{ij} + B\,, \tag{12.10}$$

where $B = t_{ij} X_1^{ij} - t_{ij} * X_1^{ij}$. Taking into account that

$$B = t_{ij} X_1^{ij} - t_{ij} * X_1^{ij} = -\frac{1}{4} \frac{\partial^2}{\partial Y_A^i \partial Y^{jA}} X_1^{ij} \tag{12.11}$$

by virtue of (11.7), one observes that B contains less traces than $D(A^{tr})$. The factorization is performed by dropping out the term $t_{ij} * X_1^{ij}$. The resulting expression $(D(A^{tr}))^{tr} + B$ contains less traces. If necessary, the procedure has to be repeated to get rid of lower traces. At the linearized level it terminates in a finite number of steps (two steps is enough for the Lorentz tracelessness condition). The resulting expression gives the covariant derivative in the quotient representation which is the on-mass-shell twisted adjoint representation.

Equivalently, one can use the factorization procedure with the quasiprojector Δ as explained in the end of Section 11. Upon application of one or another procedure for factorizing out the traces, the spin-s submodule of the twisted adjoint module forms an irreducible $o(d-1,2)$-module. The subset of the fields C in the twisted adjoint module with some fixed s matches the set of spin-s generalized Weyl zero-forms of Section 8. Not surprisingly, they form equivalent $o(d-1,2)$-modules. In particular, it can be checked that, upon an appropriate rescaling of the fields, the covariant constancy condition in the twisted adjoint representation

$$\widetilde{D}_0 C = 0 \tag{12.12}$$

reproduces (8.3) in the standard gauge. The precise form of the Λ–dependent terms in (8.3) follows from this construction. It is straightforward to compute the value of the Casimir operator in this irreducible $o(d-1,2)$-module (see also [75])

$$2T^{AB}T_{AB} = (s-1)(s+d-3)\,,$$

where $2s$ is the eigenvalue of N^{tw}. This value coincides with that for the unitary massless representations of any spin in AdS_d [76]. This fact is in agreement with the general observation [77, 51] that the representations carried by the zero-form sector in the unfolded dynamics are dual by a nonunitary Bogolyubov transform to the Hilbert space of single-particle states in the quantized theory.

13 Nonlinear field equations

We are now ready to search for nonlinear corrections to the free field dynamics. We will see that it is indeed possible to find a unique form for interactions, modulo field redefinitions, if one demands that the $sp(2)$ invariance of Section 5.2 is maintained at the nonlinear level. This condition is of crucial importance because, if the $sp(2)$ invariance was broken, then the resulting nonlinear equations might involve new tensor fields, different from the two-row rectangular Young tableaux one started with, and this might have no sense (for example, the new fields may contain ghosts). Thus, to have only usual HS fields as independent degrees of freedom, one has to require that $sp(2)$ invariance survives at the nonlinear level, or, in other words, that there should be a modified $sp(2)$ generator,

$$t_{ij}^{int} = t_{ij} + O(C)\ ,$$

that still satisfies $D(t_{ij}^{int}) = 0$, which is a deformation of the free field condition (11.11).

The construction of nonlinear corrections to the free field dynamics and the check of consistency order by order is quite cumbersome. These have been performed explicitly up to second order in the Weyl zero-forms [59, 60, 78] in terms of the spinorial formulation of the $d = 4$ HS theory. More refined methods have been developed to formulate the full dynamics of HS gauge fields in a closed form first in four dimensions [14, 15] and more recently in any dimension [16]. The latter is presented now.

13.1 Doubling of oscillators

A trick that simplifies the formulation is to introduce additional noncommutative variables Z. This allows one to describe complicate nonlinear corrections as solutions of some simple differential equations with respect to such variables. The form of these equations is fixed by formal consistency and by the existence of nonlinear $sp(2)$ generators that guarantee the correct spectrum of fields and the gauge invariance of all nonlinear terms they encode.

More precisely, this step amounts to the doubling of the oscillators $Y_i^A \to (Z_i^A, Y_i^A)$, and correspondingly one needs to enlarge the star product law. It turns out that a sensible definition is the following,

$$(f * g)(Z,Y) = \frac{1}{\pi^{2(d+1)}} \int dSdT\, e^{-2S_i^A T_A^i} f(Z+S, Y+S) g(Z-T, Y+T) \ , \tag{13.1}$$

which is an associative and regular product law in the space of polynomial functions $f(Z,Y)$, and gives rise to the commutation relations

$$[Z_i^A, Z_j^B]_* = -\epsilon_{ij}\eta^{AB} \ , \qquad [Y_i^A, Y_j^B]_* = \epsilon_{ij}\eta^{AB} \ , \qquad [Y_i^A, Z_j^B]_* = 0 \ .$$

The definition (13.1) has the meaning of a normal ordering with respect to the "creation" and "annihilation" operators $Z-Y$ and $Z+Y$, respectively. Actually, from (13.1) follows that the left star multiplication by $Z-Y$ and the right star multiplication by $Z+Y$ are equivalent to usual multiplications by $Z-Y$ and $Z+Y$, respectively. Note that Z independent functions $f(Y)$ form a proper subalgebra of the star product algebra (13.1) with the Moyal star product (11.3).

One can also check that the following formulae are true:

$$Y_i^A * = Y_i^A + \frac{1}{2}\left(\frac{\overrightarrow{\partial}}{\partial Y_A^i} - \frac{\overrightarrow{\partial}}{\partial Z_A^i}\right) \ , \qquad * Y_i^A = Y_i^A - \frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial Y_A^i} + \frac{\overleftarrow{\partial}}{\partial Z_A^i}\right) \ , \tag{13.2}$$

$$Z_i^A * = Z_i^A + \frac{1}{2}\left(\frac{\overrightarrow{\partial}}{\partial Y_A^i} - \frac{\overrightarrow{\partial}}{\partial Z_A^i}\right) \ , \qquad * Z_i^A = Z_i^A + \frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial Y_A^i} + \frac{\overleftarrow{\partial}}{\partial Z_A^i}\right) \ . \tag{13.3}$$

Furthermore, the appropriate reality conditions for the Lie algebra built from this associative star product algebra via commutators are

$$\bar{f}(Z,Y) = -f(-iZ, iY)\,, \tag{13.4}$$

where the bar denotes complex conjugation of the coefficients of the expansion of $f(Z,Y)$ in powers of Z and Y. This condition results from (5.9) with the involution $\dagger$ defined by the relations

$$(Y_i^A)^\dagger = iY_i^A \ , \qquad (Z_i^A)^\dagger = -iZ_i^A \ . \tag{13.5}$$

13.2 Klein operator

The distinguishing property of the extended definition (13.1) of the star product is that it admits the inner Klein operator

$$\mathcal{K} = \exp(-2z_i y^i) \ , \tag{13.6}$$

where

$$y_i \equiv V_A Y_i^A \ , \qquad z_i \equiv V_A Z_i^A$$

are the projections of the oscillators along V^A. Using the definitions (13.1)-(13.6), one can show that $\mathcal{K}$ (i) generates the automorphism τ as an inner automorphism of the extended star algebra,

$$\mathcal{K} * f(Z,Y) = f(\widetilde{Z}, \widetilde{Y}) * \mathcal{K} \ , \tag{13.7}$$

and (ii) is involutive,

$$\mathcal{K} * \mathcal{K} = 1 \tag{13.8}$$

(see Appendix 2.1 for a proof of these properties).

Let us note that *a priori* the star product (13.1) is well-defined for the algebra of polynomials (which means that the star product of two polynomials is still a polynomial). Thus the star product admits an ordinary interpretation in terms of oscillators as long as we deal with polynomial functions. But $\mathcal{K}$ is not a polynomial because it contains an infinite number of terms with higher and higher powers of $z_i y^i$. So, *a priori* the star product with $\mathcal{K}$ might give rise to divergencies arising from the contraction of an

infinite number of terms (for example, an infinite contribution might appear in the zeroth order like a sort of vacuum energy). What singles out the particular star product (13.1) is that this does not happen for the class of functions which extends the space of polynomials to include $\mathcal{K}$ and similar functions.

Indeed, the evaluation of the star product of two exponentials like $\mathcal{A} = \exp(A^{ij}_{AB} W_1{}^A_i W_2{}^B_j)$, where A^{ij}_{AB} are constant coefficients and the W's are some linear combinations of Y^A_i and Z^A_i, amounts to evaluating the Gaussian integral resulting from (13.1). The potential problem is that the bilinear form B of the integration variables in the Gaussian integral in $\mathcal{A}_1 * \mathcal{A}_2$ may be degenerate for some exponentials $\mathcal{A}_1$ and $\mathcal{A}_2$, which leads to an infinite result because the Gaussian evaluates $det^{-1/2}|B|$. As was shown originally in [15] (see also [79]) for the analogous spinorial star product in four dimensions, the star product (13.1) is well-defined for the class of functions, which we call *regular*, that can be expanded into a finite sum of functions f of the form

$$f(Z,Y) = P(Z,Y) \int_{M^n} d^n\tau \; \rho(\tau) \exp\Big(\phi(\tau) z_i y^i\Big) , \tag{13.9}$$

where the integration is over some compact domain $M^n \subset \mathbb{R}^n$ parametrized by the coordinates τ_i ($i = 1, \dots, n$), the functions $P(Z,Y)$ and $\phi(\tau)$ are arbitrary polynomials of (Z,Y) and τ_i, respectively, while $\rho(\tau)$ is integrable in M^n. The key point of the proof is that the star product (13.1) is such that the exponential in the Ansatz (13.9) never contributes to the quadratic form in the integration variables simply because $s_i s^i = t_i t^i \equiv 0$ (where $s_i = S^A_i V_A$ and $t_i = T^A_i V_A$). As a result, the star product of two elements (13.9) never develops an infinity and the class (13.9) turns out to be closed under star multiplication pretty much as usual polynomials. The complete proof is given in Appendix 2.2.

The Klein operator $\mathcal{K}$ obviously belongs to the regular class, as can be seen by putting $n = 1$, $\rho(\tau) = \delta(\tau + 2)$, $\phi(\tau) = \tau$ and $P(Z,Y) = 1$ in (13.9). Hence our manipulations with $\mathcal{K}$ are safe. This property can be lost however if one either goes beyond the class of regular functions (in particular, this happens when the quasiprojector Δ (11.12) is involved) or uses a different star product realization of the same oscillator algebra. For example, usual Weyl ordering prescription is not helpful in that respect.

13.3 Field content

The nonlinear equations are formulated in terms of the fields $W(Z,Y|x)$, $S(Z,Y|x)$ and $B(Z,Y|x)$, where B is a zero-form, while

$$W(Z,Y|x) = dx^\mu W_\mu(Z,Y|x) , \qquad S(Z,Y|x) = dZ^A_i S^i_A(Z,Y|x)$$

are connection one-forms, in space-time and auxiliary Z^A_i directions, respectively. They satisfy the reality conditions analogous to (13.4)

$$\bar{W}(Z,Y|x) = -W(-iZ, iY|x) \; , \quad \bar{S}(Z,Y|x) = -S(-iZ, iY|x) \; ,$$

$$\bar{B}(Z,Y|x) = -\widetilde{B}(-iZ, iY|x) \; .$$

The fields ω and C are identified with the "initial data" for the evolution in Z variables as follows:

$$\omega(Y|x) = W(0,Y|x) , \qquad C(Y|x) = B(0,Y|x) .$$

The differentials satisfy the standard anticommutation relations

$$dx^\mu dx^\nu = -dx^\nu dx^\mu \; , \quad dZ^A_i dZ^B_j = -dZ^B_j dZ^A_i \; , \quad dx^\mu dZ^A_i = -dZ^A_i dx^\mu \; ,$$

and commute with all other variables. The dependence on Z variables will be reconstructed by the imposed equations (modulo pure gauge ambiguities).

We require that all $sp(2)$ indices are contracted covariantly. This is achieved by imposing the conditions

$$[t^{tot}_{ij}, W]_* = 0 , \qquad [t^{tot}_{ij}, B]_* = 0 , \qquad [t^{tot}_{ij}, S^A_k]_* = \epsilon_{jk} S^A_i + \epsilon_{ik} S^A_j \; , \tag{13.10}$$

where the diagonal $sp(2)$ generator

$$t^{tot}_{ij} \equiv Y^A_i Y_{Aj} - Z^A_i Z_{Aj} \tag{13.11}$$

generates inner $sp(2)$ rotations of the star product algebra

$$[t^{tot}_{ij}, Y^A_k]_* = \epsilon_{jk} Y^A_i + \epsilon_{ik} Y^A_j \,, \qquad [t^{tot}_{ij}, Z^A_k]_* = \epsilon_{jk} Z^A_i + \epsilon_{ik} Z^A_j \,. \tag{13.12}$$

Note that the first of the relations (13.10) can be written covariantly as $D(t^{tot}_{ij}) = 0$, by taking into account that $d(t^{tot}_{ij}) = 0$.

13.4 Nonlinear system of equations

The full nonlinear system of equations for completely symmetric HS fields is

$$dW + W * W = 0 \,, \tag{13.13}$$

$$dB + W * B - B * \widetilde{W} = 0 \,, \tag{13.14}$$

$$dS + W * S + S * W = 0 \,, \tag{13.15}$$

$$S * B - B * \widetilde{S} = 0 \,, \tag{13.16}$$

$$S * S = -\frac{1}{2}(dZ^i_A dZ^A_i + 4\Lambda^{-1} dZ_{Ai} dZ^i_B V^A V^B B * \mathcal{K}) \,, \tag{13.17}$$

where we define

$$\widetilde{S}(dZ, Z, Y) = S(\widetilde{dZ}, \widetilde{Z}, \widetilde{Y}) \,. \tag{13.18}$$

One should stress that the twisting of the basis elements dZ^A_i of the exterior algebra in the auxiliary directions is *not* implemented via the Klein operator as in (13.7). Solutions of the system (13.13)-(13.17) admit factorization over the ideal generated by the nonlinear $sp(2)$ generators (13.22) defined in Section 13.6 as nonlinear deformations of the generators (11.4) used in the free field analysis. The system resulting from this factorization gives the nonlinear HS interactions to all orders.

The first three equations are the only ones containing space-time derivatives, via the de Rham differential $d = dx^\mu \frac{\partial}{\partial x^\mu}$. They have the form of zero-curvature equations for the space-time connection W (13.13) and the Z-space connection S (13.15) together with a covariant constancy condition for the zero-form B (13.14). These equations alone do not allow any nontrivial dynamics, so the contribution coming from (13.16) and (13.17) is essential. Note that the last two equations are constraints from the space-time point of view, not containing derivatives with respect to the x-variables, and that the nontrivial part only appears with the "source" term $B * \mathcal{K}$ in the V^A longitudinal sector of (13.17) (the first term on the right hand side of (13.17) is a constant). The inverse power of the cosmological constant Λ is present in (13.17) to obtain a Weyl tensor with Λ independent coefficients in the linearized equations in such a way that their flat limit also makes sense. In the following, however, we will again keep V normalized to 1, which means $\Lambda = -1$ (see Section 2.3).

13.5 Formal consistency

The system is formally consistent, *i.e.* compatible with $d^2 = 0$ and with associativity. A detailed proof of this statement can be found in the appendix 2.3. Let us however point out here the only tricky step. To prove the consistency, one has to show that the associativity relation $S * (S * S) = (S * S) * S$ is compatible with the equations. This is in fact the form of the Bianchi identity with respect to the Z variables, because S actually acts as a sort of exterior derivative in the noncommutative space (as will be shown in the next section). Associativity seems then to be broken by the source term $B * \mathcal{K}$ which

anticommutes with dz_i as a consequence of (13.16) and the definition (13.18), and brings in a term proportional to $dz_i dz^i dz_j$ (where $dz_i = V_A dZ_i^A$). This is not a problem however: since i is an $sp(2)$ index, it can take only two values and, as a result, the antisymmetrized product of three indices vanishes identically: $dz_i dz^i dz_j = 0$.

In a more compact way, one can prove consistency by introducing the noncommutative extended covariant derivative $\mathcal{W} = d + W + S$ and assembling eqs. (13.13)-(13.17) into

$$\mathcal{W} * \mathcal{W} \,=\, -\frac{1}{2}\, dZ_A^i dZ_i^A \,+\, 2\, dZ_{Ai} dZ_B^i V^A V^B B * \mathcal{K} \,,$$

$$\mathcal{W} * B = B * \widetilde{\mathcal{W}} \,.$$

In other words, $S * S$ is nothing but the ZZ component of an (x, Z)-space curvature, and it is actually the only component of the curvature allowed to be nonvanishing, xx and xZ being trivial according to (13.13) and (13.15), respectively. Consistency then amounts to the fact that the associativity relations $\mathcal{W} * (\mathcal{W} * \mathcal{W}) = (\mathcal{W} * \mathcal{W}) * \mathcal{W}$ and $(\mathcal{W} * \mathcal{W}) * B = \mathcal{W} * (\mathcal{W} * B)$ are respected by the nonlinear equations. Recall, however, that it was crucial for the consistency that the symplectic indices take only two values.

According to the general scheme of free differential algebras, the consistency of the nonlinear equations implies gauge symmetry under the local transformations

$$\delta \mathcal{W} = [\epsilon, \mathcal{W}]_* \,, \qquad \delta B = \epsilon * B - B * \widetilde{\epsilon} \,, \tag{13.19}$$

where $\epsilon = \epsilon(Z, Y|x)$ is $sp(2)$ invariant, *i.e.* $[t_{ij}^{tot}, \epsilon]_* = 0$, and otherwise arbitrary.

The consistency of the system, which means compatibility of the equations with the Bianchi identities, both in the x and in the Z sector (the latter case being verified by associativity of S), guarantees that the perturbative analysis works systematically at all orders.

13.6 $Sp(2)$ invariance

The $sp(2)$ symmetry is crucial for the consistency of the free system, to avoid unwanted ghost degrees of freedom. But as said before, survival of the $sp(2)$ invariance at the full nonlinear level is also very important, in the sense that it fixes the form of the nonlinear equations and prevents a mixture of unwanted degrees of freedom at the nonlinear level.

The rationale behind this is as follows. The conditions (13.10) guarantee the $sp(2)$ covariance of the whole framework. But this is not enough because one has to remove traces by factoring out terms which are themselves proportional to the $sp(2)$ generators. The third commutation relation in (13.10) makes this difficult. Indeed, it means that the operators S_i^A transform elements of the algebra proportional to t_{ij}^{tot} into t_{ij}^{tot} independent elements, *i.e.* the equations (13.13)-(13.17) do not allow a factorization with respect to the ideal generated by t_{ij}^{tot}.

To avoid this problem at the full nonlinear level one has to build proper generators

$$t_{ij}^{int} = t_{ij} + t_{ij}^1 + \dots \,,$$

where t_{ij}^1 and higher terms denote the field-dependent corrections to the original $sp(2)$ generators (5.6), such that they satisfy the $sp(2)$ commutation relations

$$[t_{ij}^{int}, t_{kl}^{int}] = \epsilon_{ik} t_{j\,l}^{int} + \epsilon_{jk} t_{il}^{int} + \epsilon_{il} t_{jk}^{int} + \epsilon_{jl} t_{ik}^{int}$$

and

$$D t_{ij}^{int} = 0 \,, \qquad [S, t_{ij}^{int}]_* = 0 \,, \qquad [B * \mathcal{K}, t_{ij}^{int}]_* = 0 \,. \tag{13.20}$$

What fixes the form of the nontrivial equations (13.16) and (13.17) is just the requirement that such nonlinearly deformed $sp(2)$ generators t_{ij}^{int} do exist. Actually, getting rid of the dZ's in (13.16) and (13.17) in the longitudinal sector, these equations read as

$$[s^i, s^j]_* = -\epsilon^{ij}(1 - 4B * \mathcal{K}) \,, \qquad \{s^i \,,\, B * \mathcal{K}\}_* \,=\, 0$$

(where $s^i \equiv V^A S^i_A$). This is just a realization [49] of the so called deformed oscillator algebra found originally by Wigner [80] and discussed by many authors [81]

$$[\hat{y}^i, \hat{y}^j]_* = \epsilon^{ij}(1+\nu\hat{k}) \,, \qquad \{\hat{y}^i, \hat{k}\}_* = 0 \,, \tag{13.21}$$

ν being a central element. The main property of this algebra is that, for any ν, the elements $\tau_{ij} = -\frac{1}{2}\{s_i, s_j\}_*$ form the $sp(2)$ algebra that rotates properly s_i

$$[\tau_{ij}, s_k]_* = \epsilon_{ik}s_j + \epsilon_{jk}s_i \,.$$

As a consequence, there exists an $sp(2)$ generator

$$\mathcal{T}_{ij} = -\frac{1}{2}\{S^A_i, S_{Aj}\}_* \,,$$

which acts on S^A_i as

$$[\mathcal{T}_{ij}, S^A_k]_* = \epsilon_{ik}S^A_j + \epsilon_{jk}S^A_i \,.$$

As a result, the difference

$$t^{int}_{ij} \equiv t^{tot}_{ij} - \mathcal{T}_{ij} \tag{13.22}$$

satisfies the $sp(2)$ commutation relation and the conditions (13.20), taking into account (13.10) and (13.13) - (13.17). Moreover, at the linearized level, where $S^A_i = Z^A_i$ as will be shown in the next section, t^{int}_{ij} reduces to t_{ij}. This means that, if nonlinear equations have the form (13.13) - (13.17), interaction terms coming from the evolution along noncommutative directions do not spoil the $sp(2)$ invariance and allow the factorization of the elements proportional to t^{int}_{ij}. This, in turn, implies that the nonlinear equations admit an interpretation in terms of the tensor fields we started with in the free field analysis. Let us also note that by virtue of (13.22) and (13.13)-(13.17) the conditions (13.20) are equivalent to (13.10).

Finally, let us mention that an interesting interpretation of the deformed oscillator algebra (13.21) is [49] that it describes a two-dimensional fuzzy sphere of a ν-dependent radius. Comparing this with the equations (13.16) and (13.17) we conclude that the nontrivial HS equations describe a two-dimensional fuzzy sphere embedded into a noncommutative space of variables Z^A and Y^A. Its radius varies from point to point of the usual (commutative) space-time with coordinates x, depending on the value of the HS curvatures collectively described by the Weyl zero-form $B(Z,Y|x)$.

13.7 Factoring out the ideal

The factorization procedure is performed analogously to the linearized analysis of Subsection 12.2 by choosing one or another tracelessness condition for representatives of the equivalence classes and then dropping the terms of the form $f = t^{int}_{ij} * g^{ij}$, $[f, t^{int}_{ij}] = 0$ as explained in more detail in Subsection (14.3).

Equivalently, one can use the quasiprojector approach of [57] exposed in Section 11. To this end one defines a nonlinear quasiprojector $\Delta^{int} = \Delta + \ldots$ as a nonlinear extension of the operator Δ (11.12), satisfying

$$[S\,, \Delta^{int}]_* = 0\,, \qquad D(\Delta^{int}) = 0 \tag{13.23}$$

and

$$\Delta^{int} * t^{int}_{ij} = t^{int}_{ij} * \Delta^{int} = 0\,. \tag{13.24}$$

The equations with factored out traces then take the form[28]

$$\Delta^{int} * (dW + W * W) = 0 \,, \tag{13.25}$$

[28]This form of the HS field equations was also considered by E. Sezgin and P. Sundell (unpublished) as one of us (MV) learned from a private discussion during the Solvay workshop.

$$\Delta^{int} * (dB + W * B - B * \widetilde{W}) = 0 \;, \tag{13.26}$$

$$\Delta^{int} * (dS + W * S + S * W) = 0 \;, \tag{13.27}$$

$$\Delta^{int} * (S * B - B * \widetilde{S}) = 0 \;, \tag{13.28}$$

$$\Delta^{int} * \left(S * S + \frac{1}{2}(dZ^i_A dZ^A_i + 4\Lambda^{-1} dZ_{Ai} dZ^i_B V^A V^B B * \mathcal{K}) \right) = 0 \;. \tag{13.29}$$

The factors of Δ^{int} here insure that all terms proportional to t^{int}_{ij} drop out. Note that Δ and, therefore, Δ^{int} do not belong to the regular class, and their star products with themselves and similar operators are ill-defined. However, as pointed out in Appendix E, since Δ and Δ^{int} admit expansions in power series in Z^A_i and Y^A_j, their products with regular functions are well-defined (free of infinities), so that the equations (13.25)-(13.29) make sense at all orders. Note that, in practice, to derive manifest component equations on the physical HS modes within this approach it is anyway necessary to choose a representative of the quotient algebra in one or another form of the tracelessness conditions as discussed in Sections 4.2 and 12.

Let us note that the idea to use the strong $sp(2)$ condition suggested in [75]

$$t^{int}_{ij} * B = B * t^{int}_{ij} = 0 \tag{13.30}$$

is likely to lead to a problem beyond the linearized approximation. The reason is that the elements satisfying (13.30) are themselves of the form $\Delta^{int} * B'$ [4, 57], which is beyond the class of regular functions even in the linearized approximation with Δ in place of Δ^{int}. From (13.17) it follows that the corresponding field S is also beyond the regular class. As a result, star products of the corresponding functions that appear in the perturbative analysis of the field equations may be ill-defined (infinite), *i.e.* imposing this condition may cause infinities in the analysis of HS equations [75]. Let us note that this unlucky situation is not accidental. A closely related point is that the elements satisfying (13.30) would form a subalgebra of the off-mass-shell HS algebra if their product existed. This is not true, however: the on-mass-shell HS algebra is a quotient algebra over the ideal $\mathcal{I}$ but not a subalgebra. This fact manifests itself in the nonexistence of products of elements satisfying (13.30) (see also [4, 57, 75]), having nothing to do with any inconsistency of the HS field equations. The factorization of the ideal $\mathcal{I}$ in (13.25)-(13.29) is both sufficient and free of infinities.

14 Perturbative analysis

Let us now expand the equations around a vacuum solution, checking that the full system of HS equations reproduces the free field dynamics at the linearized level.

14.1 Vacuum solution

The vacuum solution (W_0, S_0, B_0) around which we will expand is defined by $B_0 = 0$, which is clearly a trivial solution of (13.14) and (13.16). Furthermore, it cancels the source term in (13.17), which is then solved by

$$S_0 = dZ^A_i \, Z^i_A \;. \tag{14.1}$$

Let us point out that $dS_0 = 0$ and $\widetilde{S}_0 = S_0$ by the definition (13.18). From (14.1) and

$$[Z^A_i, f]_*(Z, Y) = -\frac{\partial}{\partial Z^i_A} f(Z, Y) \tag{14.2}$$

(see (13.3)) follows that the (twisted or not) adjoint action of S_0 is equivalent to the action of the exterior differential $d_Z = dZ_i^A \frac{\partial}{\partial Z_i^A}$ in the auxiliary space. The equation (13.15) at the zeroth order then becomes $\{W_0\,, S_0\}_* = d_Z W_0 = 0$ and one concludes that W_0 can only depend on Y and not on Z. One solution of (13.13) is the AdS connection bilinear in Y

$$W_0 = \omega_0^{AB}(x) T_{AB}(Y)\ , \tag{14.3}$$

which thus appears as a natural vacuum solution of HS nonlinear equations. The vacuum solution (14.1), (14.3) satisfies also the $sp(2)$ invariance condition (13.10).

The symmetry of the chosen vacuum solution is $hu(1|2{:}[d-1,2])$. Indeed, the vacuum symmetry parameters $\epsilon^{gl}(Z,Y|x)$ must satisfy

$$[S_0\,, \epsilon^{gl}]_* = 0\,, \qquad D_0(\epsilon^{gl}) = 0\,. \tag{14.4}$$

The first of these conditions implies that $\epsilon^{gl}(Z,Y|x)$ is Z–independent, *i.e.,* $\epsilon^{gl}(Z,Y|x) = \epsilon^{gl}(Y|x)$ while the second condition reconstructs the dependence of $\epsilon^{gl}(Y|x)$ on space-time coordinates x in terms of values of $\epsilon^{gl}(Y|x_0)$ at any fixed point x_0 of space-time. Since the parameters are required to be $sp(2)$ invariant, one concludes that, upon factorization of the ideal $\mathcal{I}$, the global symmetry algebra is $hu(1|2{:}[d-1,2])$.

Our goal is now to see whether free HS equations emerge from the full system as first order correction to the vacuum solution. We thus set

$$W = W_0 + W_1\ , \qquad S = S_0 + S_1\ , \qquad B = B_0 + B_1\ ,$$

and keep terms up to the first order in W_1, S_1, B_1 in the nonlinear equations.

14.2 First order correction

As a result of the fact that the adjoint action of S_0 is equivalent to the action of d_Z, if treated perturbatively, the space-time constraints (13.16) and (13.17) actually correspond to differential equations with respect to the noncommutative Z variables.

We begin by looking at (13.16). The zero-form $B = B_1$ is already first order, so we can substitute S by S_0, to obtain that B_1 is Z-independent

$$B_1(Z,Y) = C(Y|x)\ . \tag{14.5}$$

Inserting this solution into (13.14) just gives the twisted adjoint equation (12.12) (with $\widetilde{D}_0$ defined by (12.5)), one of the two field equations we are looking for.

Next we attempt to find S_1 substituting (14.5) into (13.17), taking into account that

$$f(Z,Y) * \mathcal{K} = \exp(-2z_i y^i)\, f(Z_i^A - V^A(z_i + y_i),\, Y_i^A - V^A(z_i + y_i))$$

(see Appendix 2.1), which one can write as

$$f(Z,Y) * \mathcal{K} = \exp(-2z_i y^i)\, f({}^{\perp}Z - {}^{\parallel}Y, {}^{\perp}Y - {}^{\parallel}Z)\ .$$

This means that $\mathcal{K}$ acts on functions of Z and Y by interchanging their respective longitudinal parts (taken with a minus sign) and multiplying them by a factor of $\exp(-2z_i y^i)$.

Looking at the ZZ part of the curvature, one can see that the V^A transversal sector is trivial at first order and that the essential Z-dependence is concentrated only in the longitudinal components. One can then analyze the content of (13.17) with respect to the longitudinal direction only, getting

$$\partial^i s_1^j - \partial^j s_1^i = 4\varepsilon^{ij} C({}^{\perp}Y - {}^{\parallel}Z)\, \exp(-2z_i y^i) \tag{14.6}$$

with $\partial^i = \frac{\partial}{\partial z_i}$ and $s_1^i = S^i_{1\,A}V^A$. The general solution of the equation $\partial_i f^i(z) = g(z)$ is $f_i(z) = \partial_i \epsilon + \int_0^1 dt\, t\, z_i g(z)$. Applying this to (14.6) one has

$$s_1^i = \partial^i \epsilon_1 - 2z^i \int_0^1 dt\, t\, C({}^\perp Y - t\,{}^\parallel Z)\exp(-2tz_k y^k)\ .$$

Analogously, in the V transverse sector one obtains that ${}^\perp S_i^A$ is pure gauge so that

$$S^i_{1\,A} = \frac{\partial}{\partial Z_i^A}\epsilon_1 - 2V_A z^i \int_0^1 dt\, t\, C({}^\perp Y - t\,{}^\parallel Z)\exp(-2tz_k y^k)\,,$$

where the first term on the r.h.s. is the Z-exact part. This term is the pure gauge part with the gauge parameter $\epsilon_1 = \epsilon_1(Z;Y|x)$ belonging to the Z-extended HS algebra. One can conveniently set $\frac{\partial}{\partial Z_i^A}\epsilon_1 = 0$ by using part of the gauge symmetry (13.19). This choice fixes the Z-dependence of the gauge parameters to be trivial and leaves exactly the gauge freedom one had at the free field level, $\epsilon_1 = \epsilon_1(Y|x)$. Moreover, let us stress that with this choice one has reconstructed S_1 entirely in terms of B_1. Note that s_1^i belongs to the regular class of functions (13.9) compatible with the star product.

We now turn our attention to the equation (13.15), which determines the dependence of W on z. In the first order, it gives

$$\partial^i W_1 = ds_1^i + W_0 * s_1^i - s_1^i * W_0\ .$$

The general solution of the equation $\frac{\partial}{\partial z_i}\varphi(z) = \chi^i(z)$ is given by the line integral

$$\varphi(z) = \varphi(0) + \int_0^1 dt\, z_i \chi^i(tz)\,,$$

provided that $\frac{\partial}{\partial z_i}\chi^i(z) = 0$ $(i = 1,2)$. Consequently,

$$W_1 = \omega(Y|x) + z^j \int_0^1 dt\,(1-t)\,e^{-2tz_i y^i} E_0^B \frac{\partial}{\partial^\perp Y^{jB}} C({}^\perp Y - t\,{}^\parallel Z)\ , \tag{14.7}$$

taking into account that the term $z_i ds_1^i$ vanishes because $z_i z^i = 0$. Again, W_1 is in the regular class. Note also that in (14.7) only the frame field appears, while the dependence on the Lorentz connection cancels out. This is the manifestation of the local Lorentz symmetry which forbids the presence of ω^{ab} if not inside Lorentz covariant derivatives.

One still has to analyze (13.13), which at first order reads

$$dW_1 + \{W_0, W_1\}_* = 0\ .$$

Plugging in (14.7), one gets

$$R_1 = O(C)\ , \qquad R_1 \equiv d\omega + \{\omega\,, W_0\}_*\,, \tag{14.8}$$

where corrections on the r.h.s. of the first equation in (14.8) come from the second term in (14.7), and prevent (14.8) from being trivial, that would imply ω to be a pure gauge solution[29]. The formal consistency of the system with respect to Z variables allows one to restrict the study of (13.13) to the physical space $Z = 0$ only (with the proviso that the star products are to be evaluated before sending Z to zero). This is due to the fact that the dependence on Z is reconstructed by the equations in such a way that if (13.13) is satisfied for $Z = 0$, it is true for all Z. By elementary algebraic manipulations one obtains the final result

$$R_1 = \frac{1}{2}E_0^A\, E_0^B\, \frac{\partial^2}{\partial\,{}^\perp Y^{Ai}\partial\,{}^\perp Y_i^B} C({}^\perp Y)\ ,$$

[29] In retrospective, one sees that this is the reason why it was necessary for the Z coordinates to be noncommutative, allowing nontrivial contractions, since corrections are obtained from perturbative analysis as coefficients of an expansion in powers of z obtained by solving for the z-dependence of the fields from the full system.

which, together with the equation for the twisted adjoint representation previously obtained, reproduces the free field dynamics for all spins (10.20) and (12.12).

14.3 Higher order corrections and factorization of the ideal

Following the same lines one can now reconstruct order-by-order all nonlinear corrections to the free HS equations of motion. Note that all expressions that appear in this analysis belong to the regular class (13.9), and therefore the computation as a whole is free from divergencies, being well defined. At the same time, the substitution of expressions like (10.24) for auxiliary fields will give rise to nonlinear corrections with higher derivatives, which are nonanalytic in the cosmological constant.

Strictly speaking, the analysis explained so far is off-mass-shell. To put the theory on-mass-shell one has to factor out the ideal $\mathcal{I}$. To this end, analogously to the linearized analysis of Section 12, one has to fix representatives of $\omega(Y|x)$ and $C(Y|x)$ in one or another way (for example, demanding $\omega(Y|x)$ to be AdS traceless and $C(Y|x)$ to be Lorentz traceless). The derived component HS equations may or may not share these tracelessness properties. Let us consider a resulting expression containing some terms of the form

$$A = A_0^{tr} + t_{ij}\, A_1^{ij}\,,$$

where A_0^{tr} satisfies the chosen tracelessness condition while the second term, with t_{ij} defined by (11.4), describes extra traces. One has to rewrite such terms as

$$A = A_0^{tr} + t_{ij}^{int} * A_1^{ij} + A' + A''\,, \tag{14.9}$$

where the third term

$$A' \equiv t_{ij}\, A_1^{ij} - t_{ij} * A_1^{ij} = -\frac{1}{4}\frac{\partial^2}{\partial Y^i_A \partial Y^{jA}} A^{ij}$$

contains less traces (we took into account (11.7)), and the fourth term

$$A'' \equiv (t_{ij}^{int} - t_{ij}) * A_1^{ij} = O(C) * A_1^{ij}$$

contains higher-order corrections due to the definitions (13.11) and (13.22). The factorization is performed by dropping out the terms of the form $t_{ij}^{int} * A_1^{ij}$ in (14.9). The resulting expression $A = A_0^{tr} + A' + A''$ contains either less traces or higher order nonlinearities. If necessary, the procedure has to be repeated to get rid of lower traces at the same order or new traces at the nonlinear order. Although this procedure is complicated, it can be done in a finite number of well-defined steps for any given type of HS tensor field and given order of nonlinearity. The process is much nicer of course within the spinorial realization available in the lower dimension cases [14, 78, 79] where the factorization over the ideal $\mathcal{I}$ is automatic.

15 Discussion

A surprising issue related to the structure of the HS equations of motion is that, within this formulation of the dynamics, one can get rid of the space-time variables. Indeed, (13.13)-(13.15) are zero curvature and covariant constancy conditions, admitting pure gauge solutions. This means that, locally,

$$\begin{aligned} W(x) &= g^{-1}(x) * dg(x)\,, \\ B(x) &= g^{-1}(x) * b * \widetilde{g}(x)\,, \\ S(x) &= g^{-1}(x) * s * g(x)\,, \end{aligned} \tag{15.1}$$

the whole dependence on space-time points being absorbed into a gauge function $g(x)$, which is an arbitrary invertible element of the star product algebra, while $b = b(Z, Y)$ and $s = s(Z, Y)$ are arbitrary x-independent elements of the star product algebra. Since the system is gauge invariant, the gauge functions disappear from the remaining two equations (13.16)-(13.17), which then encode the whole

dynamics though being independent of x. This turns out to be possible, in the unfolded formulation, just because of the presence of an infinite bunch of fields, supplemented by an infinite number of appropriate constraints, determined by consistency. As previously seen in the lower spin examples (see Section 7), the zero-form B turns out to be the generating function of all on-mass-shell nontrivial derivatives of the dynamical fields. Thus it locally reconstructs their x-dependence through their Taylor expansion which in turn is just given by the formulas (15.1). So, within the unfolded formulation, the dynamical problem is well posed once all zero-forms assembled in b are given at one space-time point x_0, because this is sufficient to obtain the whole evolution of fields in some neighborhood of x_0 (note that s is reconstructed in terms of b up to a pure gauge part). This way of solving the nonlinear system, getting rid of x variables, is completely equivalent to the one used in the previous section, or, in other words, the unfolded formulation involves a trade between space-time variables and auxiliary noncommutative variables (Z, Y). Nevertheless, the way we see and perceive the world seems to require the definition of local events, and it is this need for locality that makes the reduction to the "physical" subspace $Z = 0$ (keeping the x-dependence instead of gauging it away) more appealing. On the other hand, as mentioned in Section 13.6, the HS equations in the auxiliary noncommutative space have the clear geometrical meaning of describing embeddings of a two-dimensional noncommutative sphere into the Weyl algebra.

The system of gauge invariant nonlinear equations for all spins in AdS_d here presented can be generalized [16] to matrix-valued fields, $W \to W_\alpha{}^\beta$, $S \to S_\alpha{}^\beta$ and $B \to B_\alpha{}^\beta$, $\alpha, \beta = 1, ..., n$, giving rise to Yang-Mills groups $U(n)$ in the $s = 1$ sector while remaining consistent. It is also possible to truncate to smaller inner symmetry groups $USp(n)$ and $O(n)$ by imposing further conditions based on certain antiautomorphisms of the star product algebra [46, 16]. Apart from the possibility of extending the symmetry group with matrix-valued fields, and modulo field redefinitions, it seems that there is no ambiguity in the form of nonlinear equations. As previously noted, this is due to the fact that the $sp(2)$ invariance requires (13.16) and (13.17) to have the form of a deformed oscillator algebra.

HS models have just one dimensionless coupling constant

$$g^2 = |\Lambda|^{\frac{d}{2}-1}\kappa^2 \,.$$

To introduce the coupling constant, one has to rescale the fluctuations ω_1 of the gauge fields (*i.e.* additions to the vacuum field) as well as the Weyl zero-forms by a coupling constant g so that it cancels out in the free field equations. In particular, g is identified with the Yang-Mills coupling constant in the spin-1 sector. Its particular value is artificial however because it can be rescaled away in the classical theory (although it is supposed to be a true coupling constant in the quantum theory where it is a constant in front of the whole action in the exponential inside the path integral). Moreover, there is no dimensionful constant allowing us to discuss a low-energy expansion, *i.e.* an expansion in powers of a dimensionless combination of this constant and the covariant derivative. The only dimensionful constant present here is Λ. The dimensionless combinations $\bar{D}_\mu \equiv \Lambda^{-1/2} D_\mu$ are not good expansion entities, since the commutator of two of them is of order 1, as a consequence of the fact that the AdS curvature is roughly $R_0 \sim D^2 \sim \Lambda gg$. For this reason also it would be important to find solutions of HS field equations different from AdS_d, thus introducing in the theory a massive parameter different from the cosmological constant.

Finally, let us note that a variational principle giving the nonlinear equations (13.13)-(13.17) is still unknown in all orders. Indeed, at the action level, gauge invariant interactions have been constructed only up to the cubic order [13, 4, 82].

16 Conclusion

The main message of these lectures is that nonlinear dynamics of HS gauge fields can be consistently formulated in all orders in interactions in anti de Sitter space-time of any dimension $d \geqslant 4$. The level of generality of the analyzes covered has been restricted in the following points: only completely symmetric bosonic HS gauge fields have been considered, and only at the level of the equations of motion.

Since it was impossible to cover all the interesting and important directions of research in the modern

HS gauge theory, an invitation to further readings is provided as a conclusion. For general topics in HS gauge theories, the reader is referred to the review papers [3, 9, 10, 38, 58, 83, 84, 85]. Among the specific topics that have not been addressed here, one can mention:

(i) Spinor realizations of HS superalgebras in $d = 3, 4$ [43, 44, 46, 79], $d = 5$ [50, 7], $d = 7$ [52] and the recent developments in any dimension [57],

(ii) Cubic action interactions [13, 4, 82],

(iii) Spinor form of $d = 4$ nonlinear HS field equations [14, 3, 86],

(iv) HS dynamics in larger (super)spaces, *e.g.* free HS theories in tensorial superspaces [87, 51, 71] and HS theories in usual superspace [88, 89],

(v) Group-theoretical classification of invariant equations via unfolded formalism [90],

(vi) HS gauge fields different from the completely symmetric Fronsdal fields: *e.g.* mixed symmetry fields [91], infinite component massless representations [92] and partially massless fields [93],

(vii) Light cone formulation for massless fields in AdS_d [94],

(viii) Tensionless limit of quantized (super)string theory [95, 33].

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1 Basic material on lower spin gauge theories

1.1 Isometry algebras

By "space-time" symmetries one means symmetries of the corresponding space-time manifold $\mathcal{M}^d$ of dimension d, which may be isometries or conformal symmetries. The most symmetrical solutions of vacuum Einstein equations, with or without cosmological constant Λ, are (locally) Minkowski space-time ($\Lambda = 0$), de Sitter ($\Lambda > 0$) and Anti de Sitter ($\Lambda < 0$) spaces. In this paper, we only consider $\mathbb{R}^{d-1,1}$ and AdS_d spaces though the results generalize easily to dS_d space.

The Poincaré group $ISO(d-1,1) = \mathbb{R}^{d-1,1} \rtimes SO(d-1,1)$ has translation generators P_a and Lorentz generators M_{ab} $(a, b = 0, 1, \ldots, d-1)$ satisfying the algebra

$$[\, M_{ab}\, ,\, M_{cd}\,] = \eta_{ac}\, M_{db} - \eta_{bc}\, M_{da} - \eta_{ad}\, M_{cb} + \eta_{bd}\, M_{ca}\, , \tag{1.1}$$

$$[\, P_a\, ,\, M_{bc}] = \eta_{ab}\, P_c - \eta_{ac}\, P_b\, , \tag{1.2}$$

$$[P_a, P_b] = 0\,. \tag{1.3}$$

"Internal" symmetries are defined as transformations that commute with the translations generated by P_a and the Lorentz transformations generated by M_{ab} [1]. The relation (1.1) defines the Lorentz algebra $o(d-1,1)$ while the relations (1.2)-(1.3) state that the Poincaré algebra is a semi-direct product $iso(d-1,1) = \mathbb{R}^d \niplus o(d-1,1)$.

The algebra of isometries of the AdS_d space-time is given by the commutation relations (1.1)-(1.2) and

$$[P_a, P_b] = -\frac{1}{\rho^2}\, M_{ab}\,, \tag{1.4}$$

where ρ is proportional to the radius of curvature of AdS_d and is related to the cosmological constant via $\Lambda = -\rho^{-2}$. The (noncommuting) transformations generated by P_a are called transvections in AdS_d to distinguish them from the (commuting) translations. By defining $M_{\hat{d}\,a} = \rho\, P_a$, it is possible to collect all generators into the generators M_{AB} where $A = 0, 1, \ldots, \hat{d}$. These generators M_{AB} span $o(d-1,2)$ algebra since they satisfy the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC} M_{DB} - \eta_{BC} M_{DA} - \eta_{AD} M_{CB} + \eta_{BD} M_{CA}\,,$$

where η_{AB} is the mostly minus invariant metric of $o(d-1,2)$. This is easily understood from the geometrical construction of AdS_d as the hyperboloid defined by $X^A X_A = \frac{(d-1)(d-2)}{2}\rho^2$ which is obviously invariant under the isometry group $O(d-1,2)$. Since transvections are actually rotations in ambient space, it is normal that they do not commute. It is possible to derive the Poincaré algebra from the AdS_d isometry algebra by taking the infinite-radius limit $\rho \to \infty$. This limiting procedure is called Inönü-Wigner contraction [96].

1.2 Gauging internal symmetries

In this subsection, a series of tools used in any gauge theory is briefly introduced. One considers the most illustrative example of Yang-Mills theory, which corresponds to gauging an internal symmetry group.

Let g be a (finite-dimensional) Lie algebra of basis $\{T_\alpha\}$ and Lie bracket [,]. The structure constants are defined by $[T_\alpha, T_\beta] = T_\gamma f^\gamma{}_{\alpha\beta}$. The Yang-Mills theory is conveniently formulated by using differential forms taking values in the Lie algebra g.

- The connection $A = dx^\mu A^\alpha_\mu T_\alpha$ is defined in terms the vector gauge field A^α_μ.
- The curvature $F = dA + A^2 = \frac{1}{2}\, dx^\mu dx^\nu F^\alpha_{\mu\nu} T_\alpha$ is associated with the field strength tensor $F^\alpha_{\mu\nu} = \partial_{[\mu} A^\alpha_{\nu]} + f^\alpha{}_{\beta\gamma} A^\beta_\mu A^\gamma_\nu$.
- The Bianchi identity $dF + AF - FA = 0$ is a consequence of $d^2 = 0$ and the Jacobi identity in the Lie algebra.
- The gauge parameter $\epsilon = \epsilon^\alpha T_\alpha$ is associated with the infinitesimal gauge transformation $\delta_\epsilon A = d\epsilon + A\epsilon - \epsilon A$ which transforms the curvature as $\delta_\epsilon F = F\epsilon - \epsilon F$. In components, this reads as $\delta_{\epsilon^\alpha} A^\alpha_\mu = \partial_\mu \epsilon^\alpha + f^\alpha_{\beta\gamma} A^\beta_\mu \epsilon^\gamma$ and $\delta F^\alpha_{\mu\nu} = f^\alpha_{\beta\gamma} F^\beta_{\mu\nu} \epsilon^\gamma$.

The algebra $\Omega(\mathcal{M}^d) \otimes g$ is a Lie superalgebra, the product of which is the graded Lie bracket denoted by [,] [30]. The elements of $\Omega^p(\mathcal{M}^d) \otimes g$ are p-forms taking values in g. The interest of the algebra $\Omega(\mathcal{M}^d) \otimes g$ is that it contains the gauge parameter $\epsilon \in \Omega^0(\mathcal{M}^d) \otimes g$, the connection $A \in \Omega^1(\mathcal{M}^d) \otimes g$, the curvature $F \in \Omega^2(\mathcal{M}^d) \otimes g$, and the Bianchi identity takes place in $\Omega^3(\mathcal{M}^d) \otimes g$.

To summarize, the Yang-Mills theory is a fibre bundle construction where the Lie algebra g is the fiber, A the connection and F the curvature. The Yang-Mills action takes the form

$$S^{YM}[A^a_\mu] \propto \int_{\mathcal{M}^d} Tr[F\,{}^*F]\,,$$

[30]Here, the grading is identified with that in the exterior algebra so that the graded commutator is evaluated in terms of the original Lie bracket [,].

in which case g is taken to be compact and semisimple so that the Killing form is negative definite (in order to ensure that the Hamiltonian is bounded from below). Here * is the Hodge star producing a dual form. Note that, via Hodge star, the Yang-Mills action contains the metric tensor which is needed to achieve invariance under diffeomorphisms. Furthermore, because of the cyclicity of the trace, the Yang-Mills Lagrangian $Tr[F\,{}^*F]$ is also manifestly invariant under the gauge transformations.

An operatorial formulation is also useful for its compactness. Let us now consider some matter fields Φ living in some space V on which acts the Lie algebra g, via a representation T_α. In other words, the elements T_α are reinterpreted as operators acting on some representation space (also called module) V. The connection A becomes thereby an operator. For instance, if T is the adjoint representation then the module V is identified with the Lie algebra g and A acts as $A\cdot\Phi = [A,\Phi]$. The connection A defines the covariant derivative $D \equiv d + A$. For any representation of g, the transformation law of the matter field is $\delta\Phi = -\,\epsilon\cdot\Phi$, where ϵ is a constant or a function of x according to whether g is a global or a local symmetry. The gauge transformation law of the connection can also be written as $\delta A = \delta D = [D,\epsilon]$ because of the identity $[d,A] = dA$, and is such that $\delta(D\Phi) = -\,\epsilon\cdot(D\Phi)$. The curvature is economically defined as an operator $F = \frac{1}{2}[D,D] = D^2$. In space-time components, the latter equation reads as usual $[D_\mu, D_\nu] = F_{\mu\nu}$. The Bianchi identity is a direct consequence of the associativity of the differential algebra and Jacobi identities of the Lie algebra and reads in space-time components as $[D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] = 0$. The graded Jacobi identity leads to $\delta F = \frac{1}{2}\Big(\,[\,[D,\epsilon]\,,D\,] + [\,D,[D,\epsilon]\,]\,\Big) = [D^2\,,\,\epsilon] = [F\,,\,\epsilon]$.

The present notes make an extension of the previous compact notations and synthetic identities. Indeed, they generalize it straightforwardly to other gauge theories formulated via a nonAbelian connection, *e.g.* HS gauge theories[31].

1.3 Gauging space-time symmetries

The usual Einstein-Hilbert action $S[g]$ is invariant under diffeomorphisms. The same is true for $S[e,\omega]$, defined by (2.1), since everything is written in terms of differential forms. The action (2.1) is also manifestly invariant under local Lorentz transformations $\delta\omega = d\epsilon + [\omega,\epsilon]$ with gauge parameter $\epsilon = \epsilon^{ab}M_{ab}$, because $\epsilon_{a_1\ldots a_d}$ is an invariant tensor of $SO(d-1,1)$. The gauge formulation of gravity shares many features with a Yang-Mills theory formulated in terms of a connection ω taking values in the Poincaré algebra.

However, gravity is actually *not* a Yang-Mills theory with Poincaré as (internal) gauge group. The aim of this section is to express precisely the distinction between gauge symmetries which are either internal or space-time.

To warm up, let us mention several obvious differences between Einstein-Cartan's gravity and Yang-Mills theory. First of all, the Poincaré algebra $iso(d-1,1)$ is not semisimple (since it is not a *direct* sum of simple Lie algebras, containing a nontrivial Abelian ideal spanned by translations). Secondly, the action (2.1) cannot be written in a Yang-Mills form $\int Tr[F\,{}^*F]$. Thirdly, the action (2.1) is not invariant under the gauge transformations $\delta\omega = d\epsilon + [\omega,\epsilon]$ generated by *all* Poincaré algebra generators, *i.e.* with gauge parameter $\epsilon(x) = \epsilon^a(x)P_a + \epsilon^{ab}(x)M_{ab}$. For $d>3$, the action (2.1) is invariant only when $\epsilon^a = 0$. (For $d=3$, the action (2.1) describes a genuine Chern-Simons theory with local $ISO(2,1)$ symmetries.)

This latter fact is not in contradiction with the fact that one actually gauges the Poincaré group in gravity. Indeed, the torsion constraint allows one to relate the local translation parameter ϵ^a to the infinitesimal change of coordinates parameter ξ^μ. Indeed, the infinitesimal diffeomorphism $x^\mu \longrightarrow x^\mu + \xi^\mu$ acts as the Lie derivative

$$\delta_\xi = \mathcal{L}_\xi \equiv i_\xi d + d i_\xi\,,$$

[31] More precisely, one can take g as an infinite-dimensional Lie algebra that arises from an associative algebra with product law $*$ and is endowed with the (sometimes twisted) commutator as bracket. Up to these subtleties and some changes of notation, all previous relations hold for HS gauge theories considered here, and they might simplify some explicit checks by the reader.

where the inner product i is defined by

$$i_\xi \equiv \xi^\mu \frac{\partial}{\partial(dx^\mu)}\,,$$

where the derivative is understood to act from the left. Any coordinate transformation of the frame field can be written as

$$\delta_\xi e^a = i_\xi(de^a) + d(i_\xi e^a) = i_\xi T^a + \underbrace{\epsilon^a{}_b e^b + D^L\epsilon^a}_{=\delta_\epsilon\, e^a}\,,$$

where the Poincaré gauge parameter is given by $\epsilon = i_\xi\omega$. Therefore, when T^a vanishes any coordinate transformation of the frame field can be interpreted as a local Poincaré transformation of the frame field, and reciprocally.

To summarize, the Einstein-Cartan formulation of gravity is indeed a fibre bundle construction where the Poincaré algebra $iso(d-1,1)$ is the fiber, ω the connection and R the curvature, but, unlike for Yang-Mills theories, the equations of motion imposes some constraints on the curvature ($T^a = 0$), and some fields are auxiliary (ω^{ab}). A fully covariant formulation is achieved in the AdS_d case with the aid of compensator formalism as explained in Section 2.3.

2 Technical issues on nonlinear higher spin equations

2.1 Two properties of the inner Klein operator

In this appendix, we shall give a proof of the properties (13.7) and (13.8). One can check the second property with the help of (13.1), which in this case amounts to

$$\mathcal{K}*\mathcal{K} = \frac{1}{\pi^{2(d+1)}}\int dSdT\, e^{-2S_i^A T_A^i} e^{-2(s_i+z_i)(y^i+s^i)} e^{-2(z_i-t_i)(y^i+t^i)}\,,$$

with

$$s_i \equiv V_A S_i^A\,, \qquad t_i \equiv V_A T_i^A. \tag{2.1}$$

Using the fact that $s_i s^i = t_i t^i = 0$ and rearranging the terms yields

$$\begin{aligned}\mathcal{K}*\mathcal{K} &= \frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i}\int dSdT\, e^{-2S_i^A T_A^i} e^{-2s^i(z_i-y_i)} e^{-2t^i(z_i+y_i)}\\ &= \frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i}\int dSdT\, e^{-2S_A^i[V^A(z_i-y_i)-T_i^A]} e^{-2t^i(z_i+y_i)}\,.\end{aligned}$$

Since

$$\frac{1}{\pi^{2(d+1)}}\int dS e^{-2S_A^i(Z_i^A-Y_i^A)} = \delta(Z_i^A - Y_i^A)\,, \tag{2.2}$$

one gets

$$\mathcal{K}*\mathcal{K} = e^{-4z_i y^i}\int dT\delta(V^A(z_i-y_i)-T_i^A) e^{-2T_B^i V^B(z_i+y_i)}$$

which, using $V_B V^B = 1$, gives $\mathcal{K}*\mathcal{K} = e^{-4z_i y^i} e^{-2(-y^i z_i + z^i y_i)} = 1$. The formula (2.2) might seem unusual because of the absence of an i in the exponent. It is consistent however, since one can assume that all oscillator variables, including integration variables, are genuine real variables times (some fixed) square root of i, *i.e.* that the integration is along appropriate lines in the complex plane. One is allowed to do so without coming into conflict with the definition of the star algebra, because its elements are analytic functions of the oscillators, and can then always be continued to real values of the variables.

The proof of (13.7) is quite similar. Let us note that, with the help of (13.8), it amounts to check that $\mathcal{K}*f(Z,Y)*\mathcal{K} = f(\tilde{Z},\tilde{Y})$. One can prove, by going through almost the same steps shown above,

that

$$\mathcal{K} * f(Z,Y) = \mathcal{K}\, f(Z^i_A + V_A(y^i - z^i), Y^i_A - V_A(y^i - z^i)) \ .$$

Explicitly, one has

$$\mathcal{K} * f(Z,Y) = \frac{\mathcal{K}}{\pi^{2(d+1)}} \int dSdT e^{-2S^A_i(T^i_A + V_A(y^i - z^i))} f(Z-T, Y+T)$$

$$= \mathcal{K} \int dT\, \delta(T^i_A + V_A(y^i - z^i)) f(Z-T, Y+T)$$

$$= \mathcal{K}\, f(Z^i_A + V_A(y^i - z^i), Y^i_A - V_A(y^i - z^i)) \ ,$$

where we have made use of (2.2). Another star product with $\mathcal{K}$ leads to

$$\mathcal{K} * f(Z,Y) * \mathcal{K} =$$

$$\frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i} \int dSdT e^{-2T^i_A[V^A(z_i+y_i)+S^A_i]} e^{-2s_i(y^i - z^i)} f(Z^i_A + V_A(y^i - z^i) + S^i_A, Y^i_A - V_A(y^i - z^i) + S^i_A) \ ,$$

which, performing the integral over T and using (2.2), gives in the end

$$e^{-4z_i y^i} e^{2(z_i+y_i)(y^i-z^i)} f(Z^i_A + V_A(y^i - z^i) - V_A(z^i + y^i), Y^i_A - V_A(y^i - z^i) - V_A(z^i + y^i)) = f(\tilde{Z}, \tilde{Y}) \ .$$

2.2 Regularity

We will prove that the star product (13.1) is well-defined for the regular class of functions (13.9). This extends the analogous result for the spinorial star product in three and four dimension obtained in [15, 79].

Theorem 2.1. *Given two regular functions $f_1(Z,Y)$ and $f_2(Z,Y)$, their star product (13.1) $(f_1 * f_2)(Z,Y)$ is a regular function.*

<u>Proof</u>:

$$f_1 * f_2 = P_1(Z,Y) \int_{M_1} d\tau_1 \rho_1(\tau_1) \exp[2\phi_1(\tau_1) z_i y^i] \, * \, P_2(Z,Y) \int_{M_2} d\tau_2 \rho_2(\tau_2) \exp[2\phi_2(\tau_2) z_i y^i]$$

$$= \int_{M_1} d\tau_1 \rho_1(\tau_1) \exp[2\phi_1(\tau_1) z_i y^i] \int_{M_2} d\tau_2 \rho_2(\tau_2) \exp[2\phi_2(\tau_2) z_i y^i] \frac{1}{\pi^{2(d+1)}} \int dSdT \times$$

$$\times \exp\{-2S^A_i T^i_A + 2\phi_1(\tau_1)[s^i(z-y)_i] + 2\phi_2(\tau_2)[t^i(z+y)_i]\} P_1(Z+S, Y+S) P_2(Z-T, Y+T) \ ,$$

with s_i and t_i defined in (2.1). Inserting

$$P(Z+U, Y+U) = \exp\left[U^A_i \left(\frac{\partial}{\partial Z^A_{1\,i}} + \frac{\partial}{\partial Y^A_{1\,i}}\right)\right] P(Z_1, Y_1)\Big|_{\substack{Z_1 = Z \\ Y_1 = Y}} \ , \tag{2.3}$$

one gets

$$f_1 * f_2 = \int_{M_1 \times M_2} d\tau_1 d\tau_2 \rho_1(\tau_1) \rho_2(\tau_2) \exp\{2[\phi_1(\tau_1) + \phi_2(\tau_2)] z_i y^i\} \times$$

$$\times \int dSdT \exp\left\{-2S^i_A \left[-\phi_1(\tau_1) V^A (z-y)_i + \frac{1}{2}\left(\frac{\partial}{\partial Z^i_{1\,A}} + \frac{\partial}{\partial Y^i_{1\,A}}\right)\right]\right\} P_1(Z_1, Y_1)\Big|_{\substack{Z_1 = Z \\ Y_1 = Y}} \times$$

$$\times \exp\left\{-2T^i_A \left[S^A_i - \phi_2(\tau_2) V^A (z+y)_i + \frac{1}{2}\left(-\frac{\partial}{\partial Z^i_{2\,A}} + \frac{\partial}{\partial Y^i_{2\,A}}\right)\right]\right\} P_2(Z_2, Y_2)\Big|_{\substack{Z_2 = Z \\ Y_2 = Y}}$$

$$= \int_{M_1 \times M_2} d\tau_1 d\tau_2 \rho_1(\tau_1)\rho_2(\tau_2) \exp\{2[\phi_1(\tau_1) + \phi_2(\tau_2) + 2\phi_1(\tau_1)\phi_2(\tau_2)] z_i y^i\} \times$$

$$\times \exp\left\{ -\frac{1}{2} \left(\frac{\partial}{\partial Z^i_{1\,A}} + \frac{\partial}{\partial Y^i_{1\,A}} \right) \left(\frac{\partial}{\partial Z^i_{2\,A}} - \frac{\partial}{\partial Y^i_{2\,A}} \right) \right\} P_1[Z_1 - \phi_2(\tau_2)^{\,\|}(Z+Y), Z_2 - \phi_2(\tau_2)^{\,\|}(Z+Y)] \times$$

$$P_2(Z_2 + \phi_1(\tau_1)^{\,\|}(Z-Y), Y_2 - \phi_1(\tau_1)^{\,\|}(Z-Y))\Big|_{\substack{Z_1=Z_2=Z \\ Y_1=Y_2=Y}} ,$$

as one can check by using (2.2) (or, equivalently, Gaussian integration, the rationale for the equivalence being given in Appendix 2.1). The product of two compact domains $M_1 \subset R^n$ and $M_2 \subset R^m$ is a compact domain in R^{n+m} and P_1, P_2 are polynomials, so one concludes that the latter expression is a finite sum of regular functions of the form (13.9). $\square$

As explained in Section 13.1, it is important for this proof that the exponential in the Ansatz (13.9) never contributes to the quadratic form in the integration variables, which is thus independent of the particular choice of the functions f_1 and f_2. This property makes sure that the class of functions (13.9) is closed under star multiplication, and it is a crucial consequence of the definition (13.1).

Analogously one can easily prove that the star products $f * g$ and $g * f$ of $f(Z,Y)$ being a power series in Z_i^A and Y_i^A with a regular function $g(Z,Y)$ are again some power series. In other words, the following theorem is true:

Theorem 2.2. *The space of power series $f(Z,Y)$ forms a bimodule of the star product algebra of regular functions.*

2.3 Consistency of the nonlinear equations

We want to show explicitly that the system of equations (13.13)-(13.17) is consistent with respect to both x and Z variables.

We can start by acting on (13.13) with the x-differential d. Imposing $d^2 = 0$, one has

$$dW * W - W * dW = 0 ,$$

which is indeed satisfied by associativity, as can be checked by using (13.13) itself. So (13.13) represents its own consistency condition.

Differentiating (13.14), one gets

$$dW * B - W * dB - dB * \widetilde{W} - B * d\widetilde{W} = 0 .$$

Using (13.14), one can substitute each dB with $-W * B + B * \widetilde{W}$, obtaining

$$dW * B + W * W * B - B * \widetilde{W} * \widetilde{W} - B * d\widetilde{W} = 0 .$$

This is identically satisfied by virtue of (13.13), which is thus the consistency condition of (13.14).

The same procedure works in the case of (13.15), taking into account that, although S is a space-time zero-form, $dx^\mu dZ_i^A = -dZ_i^A dx^\mu$. Again one gets a condition which amounts to an identity because of (13.13).

Hitting (13.16) with d and using (13.14) and (13.15), one obtains

$$-W * S * B - S * B * \widetilde{W} + W * B * \widetilde{S} + B * \widetilde{S} * \widetilde{W} = 0 ,$$

which is identically solved taking into account (13.16).

Finally, (13.15) turns the differentiated l.h.s. of (13.17) into $-W * S * S + S * S * W$, while using (13.14) the differentiated r.h.s. becomes $-2\Lambda^{-1} dz_i dz^i(-W * B * \mathcal{K} + B * \widetilde{W} * \mathcal{K})$. Using (13.7), one is then able to show that the two sides of the equation obtained are indeed equal if (13.17) holds.

S being an exterior derivative in the noncommutative directions, in the Z sector the consistency check is more easily carried on by making sure that a covariant derivative of each equation does not lead to any new condition, *i.e.* leads to identities. This amounts to implementing $d_Z^2 = 0$.

So commuting S with (13.13) gives identically 0 by virtue of (13.13) itself. The same is true for (13.15), (13.14) and (13.16), with the proviso that in these latter two cases one has to take an anticommutator of the equations with S because one is dealing with odd-degree-forms (one-forms). The only nontrivial case then turns out to be (13.17), which is treated in Section 13.3.

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